Applied Quantum Mechanics

Chapter 1 problems

Problem 1.1

(a) The Sun has a surface temperature of 5800 K and an average radius 6.96×10^8 m. Assuming the mean Sun-Mars distance is 2.28×10^{11} m, what is the total radiative power per unit area incident on the upper Mars atmosphere facing the Sun?

(b) If the surface temperature of the Sun was 6800 K, by how much would the total radiative power per unit area incident on Mars increase?

Problem 1.2

(a) A positron has charge $+e$ and the same mass m_0 as a bare electron. The energy of a particle with rest mass m_0 moving at velocity υ with momentum $p = \gamma_{\text{Lorentz}} m_0 \upsilon$ is $E = \gamma_{\text{Lorentz}} m_0 c^2$ where c is the speed of light and γ_{Lorentz} is the Lorentz factor such that $\gamma_{\text{Lorentz}}^2 = c^2/(c^2 - \upsilon^2)$. Why can a single high-energy photon *not* decay into an electron and a positron?

(b) *Two* colliding real photons γ_1 and γ_2 can create particles that have mass (D. L. Burke et al. *Phys. Rev. Lett.* **79**, 1626 (1997)). Describe the conditions under which these photons decay into a positron and an electron.

Problem 1.3

Consider a lithium atom (Li) with two electrons missing.

(a) Draw an energy level diagram for the Li^{++} ion.

(b) Derive the expression for the energy (in eV) and wavelength (in nm) of emitted light from transitions between energy levels.

(c) Calculate the three longest wavelengths (in nm) for transitions terminating at $n = 2$.

(d) If the lithium ion were embedded in a dielectric with relative permittivity $\varepsilon_r = 10$, what would be the expression for the energy (in eV) and wavelength (in nm) of emitted light from transitions between energy levels.

Problem 1.4

A particle mass m confined by a real potential is described by wave function $\psi(x, t)$.

(a) Write down the expression for the average value of the particle position $\langle x \rangle$ and then make use of the Schrödinger equation to show that the average value of momentum is

$$\langle p \rangle = m\frac{\mathrm{d}}{\mathrm{d}t}\langle x \rangle = \frac{\mathrm{i}\hbar}{2}\int_{-\infty}^{\infty} x\frac{\mathrm{d}}{\mathrm{d}x}\left(\psi^*\frac{\mathrm{d}\psi}{\mathrm{d}x} - \frac{\mathrm{d}\psi^*}{\mathrm{d}x}\psi\right)\mathrm{d}x$$

(b) Evaluate the integral in part (a) and show that

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^*\left(-\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi\,\mathrm{d}x$$

so that one may identify the momentum operator as

$$\hat{p} = -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}$$

Problem 1.5

Create a simple model of a heterostructure diode that predicts current increases exponentially with increasing forward voltage bias. Explain the assumptions you make to develop the model. Under what conditions will this predicted behavior fail? By how much is voltage bias across an ideal diode increased to change current by a factor of 10 at room temperature ($T = 300$ K)? Does this represent a fundamental limit to power dissipation in electronic switching devices operating at room temperature?

Problem 1.6

Write down the Hamiltonian operator for (a) a one-dimensional simple harmonic oscillator, (b) a helium atom, (c) a hydrogen molecule, (d) a molecule with n_n nuclei and n_e electrons.

Problem 1.7

Calculate the classical velocity of the electron in the n-th orbit of a Li^{++} ion. If this electron is described as a wave packet and its position is known to an accuracy of $\Delta x = 1$ pm, calculate the characteristic time $\Delta\tau_{\Delta x}$ for the width of the wave packet to double. Compare $\Delta\tau_{\Delta x}$ with the time to complete one classical orbit. Should the electron be described as a particle or a wave?

Problem 1.8

What is the Bohr radius for an electron with effective electron mass $m_e^* = 0.021 \times m_0$ in InAs that has low-frequency relative permittivity $\varepsilon_{r0} = 14.55$. What is the effective Rydberg energy for an electron describing a hydrogenic orbit in the medium?

Problem 1.9

Because electromagnetic radiation possesses momentum it can exert a force. If completely absorbed by matter, the absorbed electromagnetic energy flux density divided by the speed of light is a pressure called radiation pressure.

(a) If the maximum radiative power per unit area incident on the upper Earth atmosphere facing the Sun is 1.4 kW m^{-2}, what is the corresponding radiation pressure?

(b) Estimate the photon flux needed to create the pressure in (a).

(c) Compare the result in (a) with the pressure due to one atmosphere.

(d) What is the normal fluctuation in pressure per unit time?

Problem 1.10

(a) As described in Section 1.17, Alice can transmit information to Bob via a quantum communication channel that uses single photons and nonorthogonal polarization states. Explain Bob's choice of test basis in Fig. 1.10.

(b) In the absence of a single photon source, optical quantum key distribution (QKD) uses light from an attenuated laser. In a particular system the probability of single photon emission per laser pulse is 0.09. There is a -3 dB coupling loss from the laser to glass fiber. The link operates with a clock rate of 1.25 GHz (bit time $\tau = 800$ ps), average optical loss in the link is -10 dB, and time jitter in the photodetector requires that only every second time interval be used for photon detection. What is the maximum sustained data rate for guaranteed secure QKD in the system?

(c) No light can pass between two linear polarizers if their respective polarizations are oriented at 90° to each other. If a third linear polarizer oriented at a 45° angle is placed between the two linear polarizers, what is the maximum fraction of incident light intensity that can pass through the system?

Problem 1.11

A particle mass m confined by a real potential is described by a wave function $\psi(x, t)$ and Schrödinger's equation. Show that

$$\frac{d}{dt}\int_{-\infty}^{\infty} |\psi(x, t)|^2 dx = 0$$

so that if the wave function $\psi(x, t)$ is normalized it remains so for all time.

Problem 1.12

A sphere has uniform charge density $\rho = e \times 1.5 \times 10^{28}\ \mathrm{m}^{-3}$.

(a) Calculate the force on a electron point-particle initially placed at the surface of the sphere.

(b) Assuming the electron in (a) is free to move in the volume of the sphere, what is the potential seen by the electron and what is its subsequent motion?

(c) What photon energy and wavelength can be absorbed by the system?

(d) At what radius R_c does an electron of mass m_0 have a predicted peak magnitude of electron velocity that exceeds the speed of light in vacuum?

(e) If the total system, consisting of the sphere of positive charge density and the electron, is charge neutral what is the value of the radius R? How does electron dispersion limit the validity of your model?

Problem 1.13

The wave function at time $t = 0$ for an electron localized as a Gaussian wave packet in one-dimension centered at $x = x_0$ and having spatial width σ_x and momentum $\hbar k_0$ is

$$\psi(x, t = 0) = A e^{ik_0 x} e^{-(x - x_0)^2/(4\sigma_x^2)}.$$

(a) Find the wave function normalization constant A and show that $\langle x\rangle = x_0$ and $\sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sigma_x$.

(b) Take the Fourier transform of $\psi(x, t = 0)$ to determine $\psi(k, t = 0)$. Making the substitution $2\sigma_k = 1/\sigma_x$, compare your result to that of part (a). Explain the significance of $\sigma_k\sigma_x = 1/2$. Hint: use Cauchy's integral theorem, which states that $\oint f(z)dz = 0$ where z is a complex variable and $f(z)$ has no poles within the closed integration loop.

(c) Find $\psi(x, t = 0)$ and $\psi(k, t = 0)$ in the limit $\sigma_x \to 0$ and describe how the electron wave packet evolves with time. If the electron's location is measured with absolute certainty at time $t = 0$, where can we expect to locate the electron at any subsequent time? Explain your result.

(d) *If* a localized single *photon* in free-space can be described by the same initial Gaussian wave packet $\psi(x, t = 0)$, how will it evolve in time? Explain the difference in the predicted behavior of the photon compared to the electron.

Problem 1.14

Optical quantum key distribution (QKD) protocol may be viewed as a sensor capable of detecting eavesdropping on an optic link. Alice and Bob use the B92 protocol and measure error rate to detect the presence of Eve, an eavesdropper, in an otherwise lossless optical QKD link.

(a) What is the average error ratio generated by eavesdropping?

(b) What can Alice and Bob infer about the methods used by Eve?

Problem 1.15

The $\pm\hbar/2$ spin of an electron charge e and mass m_0 emerges as a consequence of rotational symmetry and a relativistic treatment of the Schrödinger equation.

(a) *Suppose* spin were a classical concept due to the rotation of a spherical electron of classical radius $r_e = \dfrac{e^2}{4\pi\varepsilon_0 m_0 c^2}$ about an axis passing through its center. Assuming a uniform classical electron density in the sphere, explicitly find the electron's moment of inertia.

(b) If the angular momentum of the classical electron is $\hbar/2$ what is the speed of the surface of the electron at its equator? Compare this to the speed of light, c, and comment.

(c) If all the electron density is in a thin torus of vanishingly small cross-section making a ring at the equator, calculate the radius at which the ring speed is 1% of c. Calculate the current in the ring and the magnetic field through the ring.

(d) Calculate the magnitude and direction of the classical force due to the coulomb interaction experienced by the electron charge density of the ring in (c).

(e) What happens to the force you calculated in (d) if, as in quantum mechanics, the electron is a point particle?

Problem 1.16

Random generation of binary 1 or 0 can be physically guaranteed by quantum mechanics and so there is interest in using this as a resource for stochastic computing. Numbers can be represented as probabilities of a binary 1 or 0 signal in a clocked bit-stream of length n_{bits} such that as $n_{\text{bits}} \to \infty$ the average value of the signal is a number distributed in the interval [0,1]. For a 2×2 mixing matrix $\mathbf{A}$ and average signal vector $\mathbf{s}$, the vector $\mathbf{x} = \mathbf{As}$ may be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

so that evaluation of $\mathbf{x}$ requires both multiplication and addition. A circuit that can perform this multiply-accumulate function on stochastic data with output scaled to fall in the interval [0,1] is shown in Fig. 1.6.

(a) When evaluating the determinant $|\mathbf{A}|$ how does root-mean-square (RMS) error scale as a function of n_{bits} and two-norm condition number of the matrix?

(b) Multiplication of two stochastic bit-streams can be achieved using the AND operation and addition can be achieved using the MUX function with a random select. What circuit elements can perform (i) subtraction and (ii) division of two stochastic bit-streams?

Problem 1.17

(a) Show that a Gaussian pulse describing an electron with effective mass m_e^* moving in the x-direction at group velocity $\upsilon_g = d\omega(k_0)/dk$, central momentum $\hbar k_0 = m_e^* \upsilon_g$, and standard deviation σ_x about x_0 at time $t = 0$ is

$$\psi(x, t) = \left(\frac{\sigma_x^2}{2\pi}\right)^{1/4} \frac{1}{\sqrt{\sigma_x^2 + \frac{i\hbar t}{m_e^*}}} e^{\left(i\frac{m_e^* \upsilon_g}{\hbar}\left(x - x_0 - \frac{\upsilon_g t}{2}\right) - \frac{(x - x_0 - \upsilon_g t)^2}{(4\sigma_x^2 + i2\hbar t/m_e^*)}\right)}$$

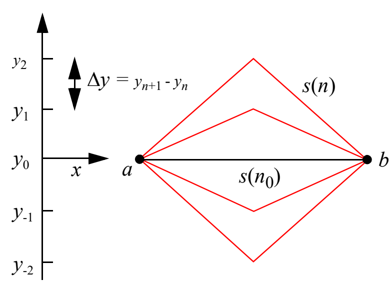
For time $0 \le t \le 0.6$ ps, find $D(t)$, the overlap area of $|\psi(x, t)|^2$ with $|\psi(x, 0)|^2$, when mean position of $|\psi(x, t)|^2$ is shifted to $x = 0$. Obtain the quantitative measure of pulse dispersion $D(t)$ using parameters $E = 50$ meV, $m_e^* = 0.07 \times m_0$ and $\sigma_x = 10$ nm.

(b) Show that the pulse dispersion measure is bounded to $0 \le D \le 1$

Problem 1.18

The shortest path length of a particle with wave properties moving in free-space in the x-direction from point a to point b is $s(n_0)$. The particle has energy $E = \hbar\omega$ and momentum $\hbar\mathbf{k}$. Assuming that at any instant $\mathbf{k}$ is parallel to the path taken $s(n)$ then, since in free space $A_n = A_0$, the total amplitude at b at time t is the sum over all possible paths

$$\mathbf{A}_{tot} = \sum_n A_n e^{iks(n) - i\omega t} = A_0 e^{-i\omega t} \sum_{n=-n_{max}}^{n=n_{max}} e^{iks(n)} = A_0 e^{-i\omega t} e^{iks(n_0)} \Phi(n)$$



For the equilateral triangle paths in the x-y plane with constant step increase in midpoint height Δy shown in the figure, plot $\mathrm{Im}(\Phi(n))$ as a function of $\mathrm{Re}(\Phi(n))$ using parameters $s(n_0) = 1$, $k = \pi$, $\Delta y = 0.005$, and $n_{max} = 2000$. Explain the results obtained.

Problem 1.19

A particle of positive non-zero energy $E = \hbar\omega$ moving unimpeded in free-space from left-to-right in the x direction is described by wave function $\psi(x, t) = A\, e^{i(kx - \omega t)}$ where A is the amplitude and k is the wave vector. The probability of finding the particle at position x is $|\psi(x, t)|^2$.

(a) Why are solutions of the form $\psi(x, t) = A \sin(kx)\, e^{-i\omega t}$ not allowed?

(b) Use Taylor expansion to find an expression to first-order for $\psi(x + \varepsilon)$ where ε is a very small displacement. Use your result to find the generator that, when acting on the wave function, results in infinitesimal spatial translation $x \to x + \varepsilon$. Comment on implications for conservation laws.