Applied Quantum Mechanics

Appendix H problems

**Problem H.1**

Find the largest and smallest volume of a convex plug manufactured from a sphere of radius $r = 1$ cm that fits exactly into a circular hole of radius $r = 1$ cm, an isosceles triangle with base 2 cm and a height $h = 1$ cm, and a half circle radius $r = 1$ cm and base 2 cm.

**Problem H.2**

An initially stationary particle mass $m_1$ is on a frictionless table surface and another particle mass $m_2$ is positioned vertically below the edge of the table. The distance from the particle mass $m_1$ to the edge of the table is $l$. The two particles are connected by a taught, light, inextensible string of length $L > l$.

(a) How much time elapses before the particle mass $m_1$ is launched off the edge of the table?

(b) What is the subsequent motion of the particles?

(c) How is your answer for (a) and (b) modified if the string has spring constant $\kappa_0$?

**Problem H.3**

The velocity of waves in shallow water may be approximated as $\upsilon = \sqrt{gh}$ where $g$ is the acceleration due to gravity and $h$ is the depth of the water. Sketch the lowest frequency standing water wave in a 5 m long garden pond that is 0.9 m deep and estimate its frequency.

**Problem H.4**

What is the dispersion relation of a wave whose group velocity is (a) half the phase velocity, (b) twice the phase velocity, (c) four times the phase velocity, and (d) the negative of phase velocity?

**Problem H.5**

(a) If *complex* field $\mathbf{G} = (\mathbf{D}/\sqrt{\varepsilon_0} + \mathrm{i}\mathbf{B}/\sqrt{\mu_0})/\sqrt{2}$ show that Maxwell's equations in free space and in the absence of free charges may written as the complex equations

$$\nabla \cdot \mathbf{G} = 0$$

and

$$\mathrm{i}\frac{\partial \mathbf{G}}{\partial t} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \nabla \times \mathbf{G}$$

where $\mathbf{G} = \frac{1}{\sqrt{2}}\left(\frac{\mathbf{D}}{\sqrt{\varepsilon_0}} + \mathrm{i}\frac{\mathbf{B}}{\sqrt{\mu_0}}\right)$

(b) Show energy flux density in the electromagnetic field given by the Poynting vector is

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \frac{-\mathrm{i}}{\sqrt{\varepsilon_0 \mu_0}}(\mathbf{G}^* \times \mathbf{G})$$

(c) If the field **G** is purely real, what is the value of **S**?

(d) Show that the electromagnetic energy density is $U = |\mathbf{G}|^2$.

(e) How would Maxwell's equations be modified if magnetic charge $g$ (magnetic monopoles) exists? Derive an expression for conservation of magnetic current and write down a generalized Lorentz force law that includes magnetic charge. Write Maxwell's equations with magnetic charge in terms of a field $\mathbf{G}$.

**Problem H.6**

(a) A thin dielectric film with relative permittivity $\varepsilon_{r_1} = 10$ uniformly coats a small metal sphere and doubles the capacitance to $2.2 \times 10^{-18}$ F. What is the thickness of the dielectric film and what is the single electron charging energy of the dielectric coated metal sphere?

(b) The dielectric coated metal sphere of part (a) is now coated with metal. What is the new value of the single-electron charging energy for the central metal sphere?

(c) Compare the result in (b) to the charging energy of a metal sphere radius $0.5\ \text{nm} < r_0 \leq 10\ \text{nm}$ embedded in a dielectric of relative permittivity $\varepsilon_r = 10$ and surrounded by metal shell of internal radius $r_1 = 2r_0$. Plot single-electron charging energy $\Delta E$ as a function of $r_0$.

**Problem H.7**

(a) A diatomic molecule has atoms with mass $m_1$ and $m_2$. An isotopic form of the molecule has atoms with mass $m'_1$ and $m'_2$. Find the ratio of vibration oscillation frequency $\omega / \omega'$ of the two molecules.

(b) What is the ratio of vibrational frequencies for carbon monoxide isotope 12 ($^{12}C^{16}O$) and carbon monoxide isotope 13 ($^{13}C^{16}O$)?

**Problem H.8**

A one centimeter long linear chain of spherical atoms has nearest neighbor spacing of 0.25 nm.

(a) What is the minimum diameter of a one atom thick disk made of these atoms?

(b) What is the minimum diameter of a sphere made of these atoms?

**Problem H.9**

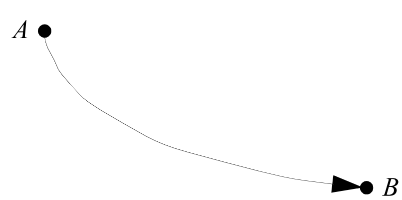
An electromagnetic wave has electric field $\mathbf{E}(\mathbf{r}, \omega) = \mathbf{E}_0(\omega)e^{i(k'(\omega) + ik''(\omega))\hat{\mathbf{k}} \cdot \mathbf{r}}$ where $k'(\omega)$ and $k''(\omega)$ are the real and imaginary parts, respectively, of the frequency-dependent wave number. The wave propagates in a homogeneous dielectric characterized by $\mu_r = 1$ and complex permittivity function $\varepsilon(\omega) = \varepsilon_0\varepsilon_r(\omega) = \varepsilon_0(\varepsilon_r'(\omega) + i\varepsilon_r''(\omega))$, where $\varepsilon_r'(\omega)$ and $\varepsilon_r''(\omega)$ are the real and imaginary parts, respectively, of the frequency-dependent relative permittivity function.

(a) Derive the expression for refractive index $n_r(\omega) = \sqrt{\frac{1}{2}(\varepsilon_r'(\omega) + \sqrt{\varepsilon_r'^2(\omega) + \varepsilon_r''^2(\omega)})}$.

(b) Introduce absorption coefficient $\alpha(\omega) = 2k''(\omega)$ and show that $\alpha(\omega) = \frac{\omega}{c}\frac{\varepsilon_r''(\omega)}{n_r(\omega)}$.

**Problem H.10**

A particle moves between two points $A$ and $B$ in a vertical plane as illustrated in the figure below. If acceleration due to gravity is $g$ and velocity is initially zero, find the shape of the frictionless surface on which the particle must move to give a trajectory that takes the shortest time.




**Problem H.11**

Materials with negative relative permeability and negative relative permittivity can display negative refractive index. In this situation group velocity is the negative of phase velocity. Suppose a point source of electromagnetic radiation in air is placed at a distance $z_1$ normal to the surface of a slab of negative refractive index material of thickness $z_2 > z_1$. The value of the negative refractive index material is $n_r = -1$. Use ray tracing to find the positions at which electromagnetic radiation from the point source is focused to a point. Comment on the statement that this slab of negative index material makes a "perfect lens".

**Problem H.12**

An electromagnetic field of wavelength $\lambda_0 = 1500$ nm in free space propagates around the inside circumference of a silica dielectric disk resonator of density $\rho = 2.2$ g cm$^{-3}$ and refractive index $n_r = 1.5$. The disk has radius $R = \frac{0.2}{2\pi}$ mm and thickness $d = 1$ μm. Electromagnetic field loss in the disk is dominated by surface roughness with average value $\alpha = 0.016$ cm$^{-1}$.

(a) Calculate the lowest natural radial mechanical oscillation frequency of the disk in the absence of light.

(b) Calculate the maximum resonant enhancement in electric field and electric field intensity and the full-width half-maximum (FWHM) in the field intensity frequency spectrum.

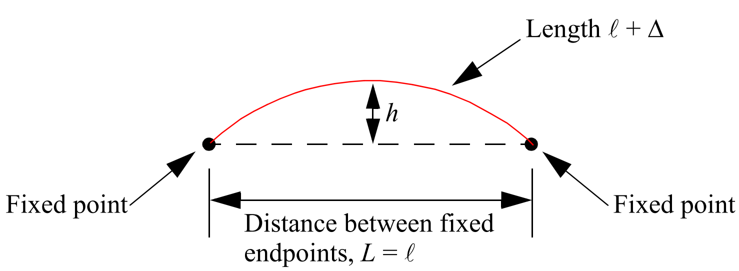
(c) Repeat the calculation in (b) but for $R = \frac{0.02}{2\pi}$ mm.

(d) Compare and explain the results obtained in (b) and (c).

(e) 10 μW of optical power at $\lambda_0 = 1500$ nm wavelength is coupled into the dielectric disk in (b). Estimate the force exerted on the disk due to radiation pressure and estimate the change in disk radius if Young's modulus for the dielectric material is 73 GPa. Compare the resulting shift in resonant frequency to the optical FWHM. What optical modulation depth might be achievable in the system?

**Problem H.13**

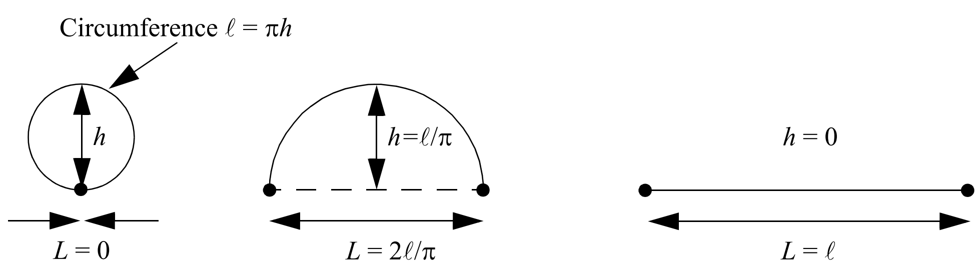
The initial length $\ell = 2$ μm of a thin horizontal micro-beam increases by $\Delta$ causing the beam to describe the arc of a circle between fixed endpoints separated by distance $L = \ell$.




(a) Calculate and plot the height of the midpoint $h$ as a function of $\Delta$ for 0 nm < $\Delta$ < 10 nm.

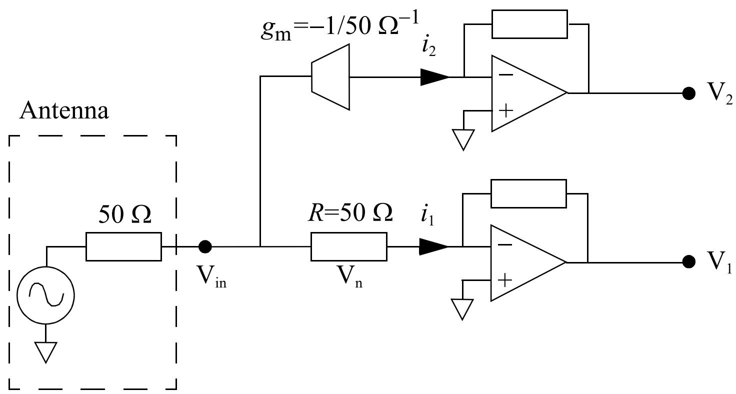
(b) Calculate and plot the mechanical gain $g \equiv h/\Delta$ for 0 nm < $\Delta$ < 10 nm.

(c) Setting $\Delta = 0$, find and plot $h$ as a function of fixed endpoint separation $L$ in the continuous range 0 to $\ell$. The following shows examples for $L = 0$, $L = 2\ell/\pi$, and $L = \ell$.



**Problem H.14**

Johnson (*Phys. Rev.* **32**, 97 (1928)) and Nyquist (*Phys. Rev.* **32**, 110 (1928)) showed that thermal fluctuations (whose cause is fundamentally due to interactions between quantized particle states) create RMS voltage noise $V_{RMS} = \sqrt{4Rk_BT\Delta f}$ in a macroscopic resistor of value *R* (ohms) at absolute temperature *T* (kelvin) measured over a frequency bandwidth $\Delta f$, so long as the frequencies considered $f \ll k_BT/(2\pi\hbar)$. This noise can limit sensitivity of a RF receiver. Bruccoleri et al. (*IEEE J. Solid-State Circuits*, **39**, 275 (2004)) showed how the following circuit, in which the current-source transconductance amplifier ($g_m$ cell) is an inverter, could be used to cancel thermal noise generated by the input load resistor *R*.




Explain how this noise cancellation works by evaluating the current $i_1$ and $i_2$ for a voltage signal $V_{in}$ at the input and voltage noise $V_n$ generated in the resistor *R*. What physical principals and conservation laws do you exploit to analyze the circuit? What limits the performance of the noise cancellation circuit?

**Problem H.15**

Suppose dipole radiation energy-loss rate $\frac{dU}{dt}$ from an electron of charge *e*, mass $m_0$, and acceleration *a*, obeys the Larmor formula

$$\frac{dU}{dt} = \frac{-2e^2a^2\varepsilon_0\mu_0}{4\pi\varepsilon_0 3c}$$

where *c* is the speed of light and $\varepsilon_0$ and $\mu_0$ is the permittivity and permiability of free-space respectively.

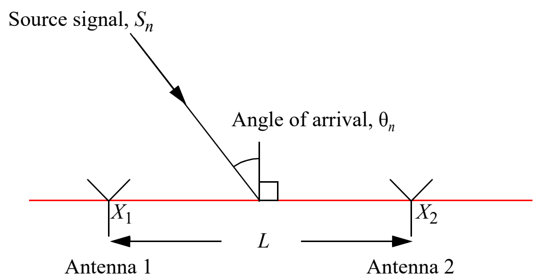
(a) *If* it is possible to describe electron motion around a proton classically and *if* the electron is initially in a circular orbit of radius *r* around the proton, what is the acceleration and velocity of the electron and how long does it take the electron to complete one round-trip assuming $r = a_B = 0.0529$ nm?

(b) What is the total (non-relativistic) energy, *U*, of the electron in (a)?

(c) If the energy loss due to electromagnetic radiation occurs slowly compared to the round-trip time, $\tau_r$, an adiabatic approximation assumes the orbit radius, *r*, remains almost circular at all times. Use the time-derivative of total electron energy at radius *r* calculated in (b) and the Larmor formula to find the time it takes the electron to radiate away all its kinetic energy and arrive at the origin, $r = 0$ (corresponding to a classical collapse of the hydrogen atom).

(d) Within the approximation of (c), at what radius is the electron velocity 10% of the speed of light? How do relativistic effects influence your calculation in (c)?

**Problem H.16**




Two identical antennas, labeled 1 and 2, are separated in free-space by distance $L$. If the antennas receive an electromagnetic signal from source $S_n$ that has angular frequency $\omega_n$ then, as a function of time $t$, a unit-amplitude source signal is $S_n(t) = e^{i\omega_n t}$. If the $n$-th signal $S_n(t)$ is a plane wave and has an angle of arrival $\theta_n$ measured anticlockwise from normal incidence then there is a relative phase difference of $\phi_n$ between the contribution of $S_n(t)$ arriving at antenna 1 and 2. In general the relationship between angle of arrival $\theta_n$ of the $n$-th signal and the phase difference $\phi_n$ is

$$\phi_n = \frac{2\pi L}{\lambda_n}\sin(\theta_n)$$

for a signal of wavelength $\lambda_n = 2\pi c/\omega_n$ ,where $c$ is the speed of light. If there are only two plane-wave sources, $S_1(t)$ and $S_2(t)$, then each antenna receives the sum of the two signals and at any given time this sum may be written in matrix form as

$$\mathbf{X} = \mathbf{AS}$$

where vector

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

describes signal $X_1(t)$ received at antenna 1 and signal $X_2(t)$ received at antenna 2,

$$\mathbf{S} = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$$

describes sources $S_1(t)$ and $S_2(t)$, and the time-independent complex mixing matrix is

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

(a) Find the matrix elements of the mixing matrix, $\mathbf{A}$.

(b) Find the inverse mixing matrix $\mathbf{A}^{-1}$ and find the conditions when it *not* possible to separate the source signals using $\mathbf{S} = \mathbf{A}^{-1}\mathbf{X}$.

(c) In a typical wireless receiver system implementation the complex signals are separated into their real (in-phase, I) and imaginary (quadrature, Q) components at each antenna relative to a reference local oscillator. This doubles the size of the mixing matrix. Find the matrix elements for the mixing matrix in this case.

(d) Discuss how the ability to separate source signals is changed if the position of the antennas and sources vary in time so that $L = L(t)$ and $\theta_n = \theta_n(t)$?

(e) Can a RF receiver be used to directly measure electromagnetic field?

**Problem H.17**

The Drude model of electrical conduction predicts a zero-frequency (DC) normal metal conductivity $\sigma_0 = e^2 n\tau/m$, where $e$ is the electron charge, $n$ the electron density, $m$ the electron mass, and the characteristic electron collision time is $\tau$. The frequency dependent (AC) conductivity is

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau} = \frac{\omega_p^2 \varepsilon_0 \tau}{1 - i\omega\tau}$$

where $\omega_p = 2\pi f_p$ is the electron plasma frequency and $1/\tau = \omega_\tau = 2\pi f_\tau$ is the electron collision rate. A linearly polarized electromagnetic plane-wave propagating in the $x$-direction and normally incident on a planar metal interface at $x = x_0$ has electric field $E_x(x) = E_{x0} e^{ikx}$ in the metal, where

$$k = \frac{\omega}{c}\sqrt{\varepsilon_r(\omega)} = \frac{\omega}{c}\sqrt{1 - \frac{\sigma_0 \tau}{\varepsilon_0(1 + \omega^2\tau^2)} + i\frac{\sigma_0}{\varepsilon_0 \omega(1 + \omega^2\tau^2)}}$$

and $\varepsilon_r(\omega)$ is the permittivity of the metal. For copper $\sigma_0(\mathrm{Cu}) = 5.9 \times 10^7\ \mathrm{S\ m^{-1}}$, $n(\mathrm{Cu}) = 8.46 \times 10^{28}\ \mathrm{m^{-3}}$, $\tau(\mathrm{Cu}) = 25\ \mathrm{fs}$, $f_p(\mathrm{Cu}) = 2600\ \mathrm{THz}$, and $f_\tau(\mathrm{Cu}) = 6.4\ \mathrm{THz}$.

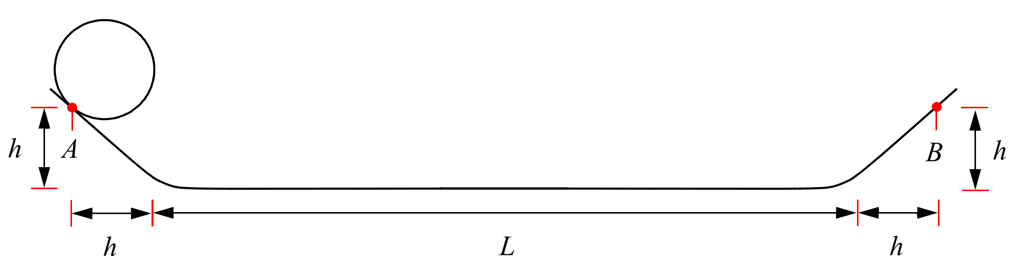
(a) Find an expression for the frequency dependent skin depth $\delta_x(\omega) = 1/(\mathrm{Im}(k(\omega))$ of copper in the low-frequency limit, $\omega\tau \ll 1$. What is the value of $\delta_x$ for an electromagnetic field oscillating at frequency $f = 100\ \mathrm{GHz}$? How does the value of surface resistance $R_s = 1/(\sigma_0 \delta_x)$ vary in the frequency range $1\ \mathrm{GHz} < f < 1\ \mathrm{THz}$?

(b) Find an expression for the skin depth of copper in the frequency range $f_\tau < f < f_p$. What is the value of $\delta_x$ at frequency $f = 100\ \mathrm{THz}$ and how does it vary with frequency in the range $10\ \mathrm{THz} < f < 1000\ \mathrm{THz}$?

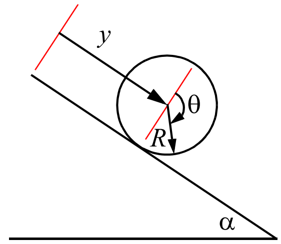
(c) What happens to the propagation of electromagnetic field when $f > f_p$?

**Problem H.18**

(a) A uniform disk of mass $m$, initially at rest and with point-of-contact on a frictionless surface at position $A$, moves down the indicated surface in the presence of acceleration due to gravity. How long does it take the point of contact to reach point $B$? What value of $h/L$ minimizes the time of travel from $A$ to $B$?




(b) Suppose there is just enough friction to ensures a uniform disk radius $R$ and mass $m$ rolls down an inclined plane set at angle $\alpha$ without slipping. Show that its acceleration is 2/3 of the value it would have if there were no friction.




(c) How far does the disk travel on the surface (a) in the presence of friction described in (b)?

**Problem H.19**

In fixed Cartesian coordinates a single particle with position coordinates $x_i$ ($i = 1, 2, 3$) has kinetic energy that is only a function of the time derivative $\dot{x}_i = dx_i/dt$ giving a Lagrangian defined as the difference between kinetic energy, $T = T(\dot{x}_i)$, and potential energy, $V = V(x_i)$ so that

$$\mathcal{L}(x_i, \dot{x}_i) = T - V \tag{1.1}$$

Hamilton's principle that states the path followed by a dynamical system moving from one point to another in configuration space within a given time interval, $t_1$ to $t_2$, minimizes the time integral of the Lagrangian. In the formalism of calculus of variations an extremum exists if the action integral

$$\delta\int_{t_1}^{t_2}\mathcal{L}(x_i, \dot{x}_i)dt = 0 \tag{1.2}$$

This action integral finds stationary action. Show, using calculus of variations, that the Lagrange equations of motion

$$\frac{\partial\mathcal{L}}{\partial x_i} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}_i} = 0 \tag{1.3}$$

may be obtained from the extremum of the action integral describing a dynamical system moving from one point to another in configuration space within a given time interval, $t_1$ to $t_2$.

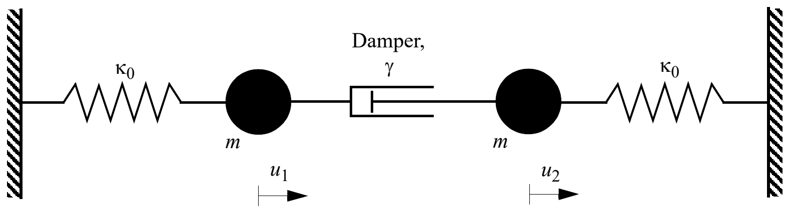
**Problem H.20**

A conservative system of $N$ particles and $n \le 3N$ degrees of freedom has particle positions $x_i = x_i(q_1, q_2, \ldots, q_n)$ and $i = 1, 2, \ldots, 3N$ where the $q_i$ form a proper set of independent generalized coordinates. Show that

$$\frac{\partial\mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i} = 0$$

**Problem H.21**

Two identical oscillators with motion in one-dimension, each of mass $m$ and spring constant $\kappa_0$, are coupled by a damping piston with velocity-dependent friction force $-\gamma m(du_1/dt - du_2/dt)$, where $\gamma$ is a damping rate and $u_{1,2}$ is particle displacement from equilibrium.




(a) Find the equations of motion and the eigenfrequencies for displacements $u_1$ and $u_2$.

(b) Solve for transient evolution of particle position when $\kappa_0/m = 1$, $\gamma = 0.1$, and the initial condition is $u_1 = 2$ and $u_2 = 1$ at time $t = 0$.

(c) Explain the motion remaining in the system in the limit $t \to \infty$. What percentage of initial energy is dissipated in the limit $t \to \infty$?

**Problem H.22**

The standard forward finite-difference first derivative of a smooth real-valued function of real variable $x$ is

$$f'(x) = \frac{f(x+h_0) - f(x)}{h_0}$$

This loses accuracy due to machine rounding error as $h_0$ becomes small. If $u$ is the unit roundoff (e.g. $u \sim 10^{-16}$ for double precision) then $h_0 \sim \sqrt{u}$ minimizes both truncation error and rounding error. This difficulty using the finite difference approximation may be avoided by considering the analytic function of complex variable $z$ that is real on the real axis with $f(z) = f(x + \mathrm{i}h_0)$.

(a) For small $h_0$ expand $f(x + \mathrm{i}h_0)$ in a Taylor series about $x$ and show that

$$f'(x) = \mathrm{Im}\left(\frac{f(x+\mathrm{i}h_0)}{h_0}\right) + err_1$$

where $err_1$ is the truncation error. Find the expression for $err_1$.

(b) Use the same approach to find the second derivative, $f''(x)$, and truncation error, $err_2$.

(c) For the function $f(x) = \mathrm{e}^x/\sqrt{\cos^3(x) + \sin^3(x)}$ compute $f'(x = \pi/4)$ by the complex method and the finite difference method. Use MATLAB's Symbolic toolbox to find the actual derivative $f'(x = \pi/4)$. Compare and plot the errors in both methods for the range $10^{-1} < h_0 < 10^{-16}$.

**Problem H.23**

A constant voltage source (a battery) is connected for a time, $t$, to a series resistance, $R$, and capacitance, $C$, circuit.

(a) What is the electrical energy stored in the capacitor, $E_{\mathrm{cap}}(t)$, and the energy delivered by the battery, $E_{\mathrm{bat}}(t)$? What is the energy efficiency, $\eta(t) = \dfrac{E_{\mathrm{cap}}(t)}{E_{\mathrm{bat}}(t)}$, and its upper bound? What is the mechanism for energy dissipation in the system as $R \to 0$?

(b) Repeat (a) for the case of a constant current source and comment on your result.

**Problem H.24**

The refractive index, $n_r$, of a linear medium with complex relative permittivity $\varepsilon_r = \varepsilon_{r,\,Re} + i\varepsilon_{r,\,Im}$ and complex relative permeability $\mu_r = \mu_{r,\,Re} + i\mu_{r,\,Im}$ satisfies $n_r^2 = \varepsilon_r\mu_r$ and so there are two possible solutions, $n_r = \pm\sqrt{\varepsilon_r\mu_r}$. Using the fact that a medium with a *small positive* imaginary part in $\varepsilon_r$ and $\mu_r$ is absorbing, show that the negative sign for the square root must be taken and that $n_r = \sqrt{\varepsilon_r}\sqrt{\mu_r}$.

**Problem H.25**

Thermal fluctuations in electron density in a resistor can cause a random instantaneous voltage drop across a resistor. In thermal equilibrium, the voltage fluctuations depend on absolute temperature *T*, frequency *f*, and the value of the resistance, *R*. Show that when $f \gg k_B T/2\pi\hbar$ there is an exponential reduction in thermal noise with increasing frequency.