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ON INITIAL-BOUNDARY VALUE PROBLEMS FOR THE NONLINEAR SCHRÖDINGER EQUATION AND THE GINZBURG-LANDAU EQUATION

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THESIS

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ABSTRACT

There are five chapters in this thesis. Well-posedness of the forced nonlinear Schrödinger equation (NLS) is shown in Chapter 1. The global solution to an initial-boundary value problem for the NLS is proved in Chapter 2. Global existence of the full-line problem for the Ginzburg-Landau equation (GL) is shown in Chapter 3. In Chapter 4, the following results concerning the half-line problem for the Ginzburg-Landau equation are established: 1) local existence-uniqueness; 2) small amplitude solution; 3) criteria for global existence. In Chapter 5, the weak solution to an initial-boundary value problem for the GL equation is obtained via Galerkin's method.

To My Wife

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CONTENTS

Introduction.

OVERVIEW OF APPROACHES TO FORCED PROBLEMS

Many forced problems occur when an external force is applied to the time evolution of systems governed by nonlinear partial differential equations. For example, in ionospheric modification experiments, one directs a radio frequency wave at the ionosphere. At the reflection point of the wave, a sufficient level of electron plasma waves is excited to make nonlinear behavior important. This is described by the nonlinear Schrödinger equation (NLS) with a nonlinear boundary value being specified [32]. We consider so-called forced problems in terms of nonlinear boundary value problems. The forced 1D nonlinear Schrödinger equation reads as follows:

$$iu_t = u_{xx} + k|u|^2 u, \qquad 0 \le x, t < \infty$$

$$u(x,0) = u_0(x), u(0,t) = Q(t), u_0(0) = Q(0)$$

with k real and $u \to 0$ as $x \to \infty$.

Forced NLS has been investigated via numerical method (e.g. Kaup [32]). It was shown that a smooth Gaussian forcing of the NLS amplitude creates a number of solitons roughly proportional to the area of the forcing amplitude. On the other hand, it has also been studied via the inverse scattering transform (IST) technique. (See Ablowitz, Carroll, Fokas in [10,12,14,15,16,17,23,24,26].) The main thrust is to determine the time evolution of spectral data so that the problem can be solved. By inverse scattering one obtains, depending on the choice of the scattering data, various complicated nonlinear singular intergro-differential equations for the time evolution of the scattering data uniquely defined in terms of the boundary condition. For the special case of a homogenous boundary condition, the scattering data are found in closed form [25]. Further, the framework and technique used to deal with NLS also extend to general AKNS systems [14]. But IST approach does not provide classical solutions (e.g. in H^2 space). Nor does it discuss global existence, an important concept in the theory of partial differential equations. There is a third approach to study the forced problems, i.e. the PDE method. For example, PDE solution of the forced Korteweg de-Vries equation (KdV) in a quarter plane was obtained by Bona and Winther in [8] and continuous dependency results were shown in [9] via analytic techniques. For the forced NLS when $u(0,t)$ is given it was shown by Carroll and Bu in [18] that for $u_0 \in H^2[0,\infty), Q \in C^2[0,\infty)$ there exists a unique global classical solution in $C^0(H^2) \cap C^1(L^2)$. Finally, initial value problems for forced linear and nonlinear partial differential equations can be considered where the forcing is assumed to be rapid compared to the unforced dynamics. In this case, a multi- scale perturbation method could be used to derive solutions in the form of asymptotic expansions [48].

In this thesis, we first in Chapter 1 show the well-posedness result for the forced NLS. Then in Chapter 2, we prove the global existence theorem for the forced NLS with different boundary condition when $u_x(0,t) + \alpha u(0,t)$ is given (here α is real). In Chapter 3, we turn to the Cauchy problem for the 1D Ginzburg-Landau equation and show the global existence for $\nu, \kappa > 0$. In Chapter 4, we study the forced Ginzburg-Landau equation. Local existence and small amplitude solution for $\nu, \kappa > 0$, partial results on global existence when $|\beta| \leq \sqrt{3}\kappa$ or $\alpha\beta > 0$ are obtained. Finally in Chapter 5, we show that there exists a weak solution to the forced Ginzburg-Landau equation with $\nu, \kappa > 0$ posed on a finite domain.

Chapter 1.
ON WELL-POSEDNESS OF THE FORCED NONLINEAR SCHRÖDINGER EQUATION (NLS)

§1.1 INTRODUCTION.

In this chapter, we study the well-posedness of the following initial-boundary value problem for the nonlinear Schödinger equation (NLS):

$$iu_t = u_{xx} + k|u|^2u \tag{1.1.1}$$

$$u(x,0) = u_0(x), u(0,t) = Q(t)$$

with $u_0(0) = Q(0)$ and k real. It was shown in [18] that for $u_0 \in H^2[0,\infty), Q \in C^2[0,\infty)$ there exists a unique global classical solution of (1.1.1) in $C^0(H^2) \cap C^1(L^2)$. Thus in §1.2 we give some growth estimates for $\|u\|_2$ and $\|u'\|_2$ which are critical to prove the well-posedness. In §1.3, for $u_0 \in H^2[0,\infty), Q \in C^2[0,\infty)$ we show the well-posedness of the forced NLS by using semigroup techniques and estimates.

We shall utilize the following notation throughout:

$$P(t) = u_x(0,t) \tag{1.1.2}$$

$$\|u'\|_2 = [\int_0^\infty |u_x(x,t)|^2 dx]^{\frac{1}{2}} \tag{1.1.3}$$

$$\|u\|_4 = [\int_0^\infty |u(x,t)|^4 dx]^{\frac{1}{4}} \tag{1.1.4}$$

$$(1.1.5) \qquad \|u\|_{m,p} = [\int_0^\infty \sum_{|\alpha|\leq m} |D_x^\alpha u(x,t)|^p dx]^{\frac{1}{p}}$$

$$(1.1.6) \qquad q = \|Q\|_{C^1[0,T]} = \sup_{0\leq t\leq T} [|Q(t)| + |Q'(t)|]$$

Further, we shall consider (1.1.1) for $0 \leq t \leq T$ and $q < R, \|u_0\|_{1,2} < R$ ($R, T < \infty$, arbitrary) unless stated otherwise.

§1.2 ESTIMATES FOR $\|u\|_2$ AND $\|u'\|_2$.

The following were established in [18].

$$(1.2.1) \qquad \|u\|_2^2 = \|u_0\|_2^2 - 2Im \int_0^t P(\tau)\overline{Q(\tau)}d\tau$$

$$(1.2.2) \qquad \|u'\|_2^2 - \frac{k}{2}\|u\|_4^4 = \|u_0'\|_2^2 - \frac{k}{2}\|u_0\|_4^4 - 2Re \int_0^t P(\tau)\overline{Q'(\tau)}d\tau$$

$$(1.2.3) \qquad \int_0^\infty u\bar{u}'dx = \int_0^\infty u_0\bar{u}_0'dx - \int_0^t Q(\tau)\overline{Q'(\tau)}d\tau + i\int_0^t |P(\tau)|^2 d\tau + \frac{k}{2}i\int_0^t |Q(\tau)|^4 d\tau$$

$$(1.2.4) \qquad \|u\|_4^4 \leq \lambda \|u'\|_2 \|u\|_2^3$$

(see [50] for (1.2.4)). By (1.2.1), (1.2.2) and (1.2.3) one has the following three estimates:

$$(1.2.5) \qquad \|u\|_2^2 \leq \|u_0\|_2^2 + 2q\sqrt{t}(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}} \leq R^2 + 2R\sqrt{t}(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}}$$

$$(1.2.6)\qquad \|u'\|_2^2 \le \frac{|k|}{2}\|u\|_4^4 + \|u_0'\|_2^2 + \frac{|k|}{2}\|u_0\|_4^4 + 2q\sqrt{t}(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}}$$

$$\le \lambda\frac{|k|}{2}\|u'\|_2\|u\|_2^3 + \|u_0'\|_2^2 + \lambda\frac{|k|}{2}\|u_0'\|_2\|u_0\|_2^3 + 2R\sqrt{t}(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}}$$

$$\le \lambda\frac{|k|}{2}\|u'\|_2\|u\|_2^3 + R^2 + c_0R^4 + 2R\sqrt{t}(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}}$$

$$(1.2.7)\qquad \int_0^t |P(\tau)|^2 d\tau \le \frac{|k|}{2}\int_0^t |Q(\tau)|^4 d\tau + |\int_0^t Q(\tau)\overline{Q'(\tau)}d\tau|$$

$$+|\int_0^\infty u\bar{u}'dx| + |\int_0^\infty u_0\bar{u}_0'dx|$$

$$\le \frac{|k|}{2}Tq^4 + Tq^2 + \|u\|_2\|u'\|_2 + \frac{1}{2}\|u_0\|_{1,2}^2$$

$$\le \frac{|k|}{2}TR^4 + TR^2 + \|u\|_2\|u'\|_2 + \frac{1}{2}\|u_0\|_2^2 + \frac{1}{2}\|u_0'\|_2^2$$

$$\le \frac{|k|}{2}TR^4 + TR^2 + \frac{1}{2}\|u\|_2^2 + \frac{1}{2}\|u'\|_2^2 + \frac{1}{2}R^2$$

From (1.2.5) and (1.2.7) one has ($q < R$)

$$\|u\|_2^2 \le R^2 + 2R\sqrt{t}(TR^4\frac{|k|}{2} + TR^2 + \|u\|_2\|u'\|_2 + \frac{1}{2}R^2)^{\frac{1}{2}}$$

$$\le R^2 + 2R\sqrt{t}(\sqrt{T\frac{|k|}{2}}R^2 + \sqrt{T}R + \|u\|_2^{\frac{1}{2}}\|u'\|_2^{\frac{1}{2}} + \sqrt{\frac{1}{2}}R)$$

$$\le R^2 + 2R^3T\sqrt{\frac{|k|}{2}} + 2TR^2 + 2R\sqrt{t}\|u\|_2^{\frac{1}{2}}\|u'\|_2^{\frac{1}{2}} + \sqrt{2}R^2\sqrt{T}$$

$$\le \hat{c} + 2R\sqrt{t}\|u\|_2^{\frac{1}{2}}\|u'\|_2^{\frac{1}{2}}$$

If $\hat{c} \ge 2R\sqrt{t}\|u\|_2^{\frac{1}{2}}\|u'\|_2^{\frac{1}{2}}$ then $\|u\|_2^2 \le 2\hat{c}$. Otherwise, $\|u\|_2^2 \le 4R\sqrt{t}\|u\|_2^{\frac{1}{2}}\|u'\|_2^{\frac{1}{2}}$. In any event,

$$(1.2.8)\qquad \|u\|_2^2 \le \max\{2\hat{c}, 4R\sqrt{t}\|u\|_2^{\frac{1}{2}}\|u'\|_2^{\frac{1}{2}}\}$$

Now if $2\hat{c} \geq 4R\sqrt{t}\|u\|_2^{\frac{1}{2}}\|u'\|_2^{\frac{1}{2}}$ then by (1.2.8) one has $\|u\|_2 \leq \sqrt{2\hat{c}}$. Otherwise, $\|u\|_2^2 \leq 4R\sqrt{t}\|u\|_2^{\frac{1}{2}}\|u'\|_2^{\frac{1}{2}}$ again by (1.2.8). Thus $\|u\|_2 \leq (4R\sqrt{t}\|u'\|_2^{\frac{1}{2}})^{\frac{2}{3}}$. In summary,

$$(1.2.9) \qquad \|u\|_2 \leq \max\{\sqrt{2\hat{c}}, (4R\sqrt{t}\|u'\|_2^{\frac{1}{2}})^{\frac{2}{3}}\} = \max\{\sqrt{2\hat{c}}, (16R^2t\|u'\|_2)^{\frac{1}{3}}\}$$

Now put (1.2.5) in the RHS of (1.2.7) to obtain ($t \leq T$)

$$\int_0^t |P(\tau)|^2 d\tau \leq \frac{|k|}{2}TR^4 + TR^2 + \frac{1}{2}\|u'\|_2^2 + \frac{1}{2}(R^2 + 2R\sqrt{T}(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}}) + \frac{1}{2}R^2$$

This implies that (by completing the square)

$$[(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}} - \frac{1}{2}R\sqrt{T}]^2 \leq \frac{5R^2T}{4} + TR^4\frac{|k|}{2} + R^2 + \frac{1}{2}\|u'\|_2^2$$

$$(1.2.10) \qquad (\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}} \leq 2R\sqrt{T} + R^2\sqrt{T\frac{|k|}{2}} + R + \frac{\|u'\|_2}{\sqrt{2}}$$

Now one can use (1.2.6) and (1.2.10) to get (noting $q < R$)

$$(1.2.11) \qquad \|u'\|_2^2 \leq \lambda\frac{|k|}{2}\|u'\|_2\|u\|_2^3 + R^2 + c_0R^4$$

$$+2R\sqrt{T}(2R\sqrt{T} + R^2\sqrt{T\frac{|k|}{2}} + R + \frac{\|u'\|_2}{\sqrt{2}})$$

$$\leq \lambda\frac{|k|}{2}\|u'\|_2\|u\|_2^3 + R^2 + c_0R^4 + 4R^2T + \sqrt{2|k|}TR^3 + 2R^2\sqrt{T} + \sqrt{2T}R\|u'\|_2$$

$$\leq \lambda\frac{|k|}{2}\|u'\|_2\|u\|_2^3 + \tilde{c} + \frac{1}{2}\|u'\|_2^2$$

The last line is obtained via $\sqrt{2T}R\|u'\|_2 \leq \frac{4TR^2}{2} + \frac{\|u'\|_2^2}{2}$ and

$$\tilde{c} = R^2 + c_0R^4 + 4R^2T + \sqrt{2|k|}TR^3 + 2R^2\sqrt{T} + \frac{4TR^2}{2}$$

Thus by (1.2.9), (1.2.11) becomes

$$(1.2.12) \qquad \|u'\|_2^2 \le \lambda|k|\|u'\|_2\|u\|_2^3 + 2\tilde{c}$$

$$\le 2\tilde{c} + \max\{\lambda|k|\|u'\|_2(2\hat{c})^{\frac{3}{2}}, \lambda|k|\|u'\|_2^2 16R^2t\}$$

If one sets $0 \le t \le \frac{1}{32\lambda|k|R^2}$ then (1.2.12) becomes

$$(1.2.13) \qquad \|u'\|_2^2 \le 2\tilde{c} + \max\{\lambda|k|\|u'\|_2(2\hat{c})^{\frac{3}{2}}, \lambda|k|\|u'\|_2^2 16R^2\frac{1}{32\lambda|k|R^2}\}$$

$$= 2\tilde{c} + \max\{\lambda|k|\|u'\|_2(2\hat{c})^{\frac{3}{2}}, \frac{1}{2}\|u'\|_2^2\}$$

If $\lambda|k|\|u'\|_2(2\hat{c})^{\frac{3}{2}} \ge \frac{1}{2}\|u'\|_2^2$ then

$$\|u'\|_2^2 \le 2\tilde{c} + \lambda|k|\|u'\|_2(2\hat{c})^{\frac{3}{2}} \le 2\tilde{c} + \frac{\lambda^2k^2(2\hat{c})^3}{2} + \frac{\|u'\|_2^2}{2}$$

hence $\|u'\|_2^2 \le 4\tilde{c} + \lambda^2k^2(2\hat{c})^3$. If $\lambda|k|\|u'\|_2(2\hat{c})^{\frac{3}{2}} \le \frac{1}{2}\|u'\|_2^2$ then $\|u'\|_2^2 \le 2\tilde{c} + \frac{1}{2}\|u'\|_2^2$ hence $\|u'\|_2^2 \le 4\tilde{c}$. Then we get

$$\|u'\|_2^2 \le 4\tilde{c} + \lambda^2k^2(2\hat{c})^3$$

for $0 \le t\frac{1}{32\lambda|k|R^2}$. Thus by taking the square root,

$$(1.2.14) \qquad \|u'\|_2 \le \sqrt{4\tilde{c} + \lambda^2k^2(2\hat{c})^3} = \lambda_1$$

From (1.2.9), noting $0 \le t \le \frac{1}{32\lambda|k|R^2}$, one has

$$(1.2.15) \qquad \|u\|_2 \le \max\{\sqrt{2\hat{c}}, (16R^2t\|u'\|_2)^{\frac{1}{3}}\} \le \sqrt{2\hat{c}} + (16R^2t\|u'\|_2)^{\frac{1}{3}}$$

$$\leq \sqrt{2\hat{c}} + (\frac{1}{2\lambda|k|}\lambda_1)^{\frac{1}{8}} = \delta_1$$

Now by (1.2.14) and (1.2.15)

$$\|u\|_{1,2} \leq \lambda_1 + \delta_1 = \tilde{\lambda}_1 \tag{1.2.16}$$

Without loss of generality one can assume that $R \leq \tilde{\lambda}_1$ in (1.2.16) otherwise one simply let $\tilde{\lambda}_1 + R$ be replaced by $\tilde{\lambda}_1$.

Now let $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$ such that $|t_{i+1} - t_i| \leq \frac{1}{32\lambda|k|R^2}$. For $t_0 \leq t \leq t_1$ one has $\|u\|_{1,2} \leq \tilde{\lambda}_1$. For $t_1 \leq t \leq t_2$ one can repeat the above process to get $\|u\|_{1,2} \leq \tilde{\lambda}_2$ and again without loss of generality we can assume that $R \leq \tilde{\lambda}_1 \leq \tilde{\lambda}_2$. Thus by induction one concludes (after $N = [\frac{T}{32\lambda|k|R^2}] + 1$ times)

PROPOSITION 1.2.1. *For $0 \leq t \leq T$ one has the following estimate $\|u\|_{1,2} \leq \tilde{\lambda}_N(R)$ for $\|u_0\|_{1,2} \leq R, \|Q\|_{C^1[0,T]} \leq R$.*

COROLLARY 1.2.2. *By estimates in [50] one obtains $\|u\|_\infty \leq \tilde{\lambda}\|u'\|_2^{\frac{1}{2}}\|u\|_2^{\frac{1}{2}} \leq \tilde{\lambda}\tilde{\lambda}_N$*

§1.3 WELL-POSEDNESS RESULTS.

Throughout this section we shall assume u, v solve (1.1.1) with data (Q, u_0) and (Q_1, v_0) both lies in $C^2[0,T] \times H^2[0,\infty) = X$. According to global existence-uniqueness theorem [18], the map

$$f : X \to Y = C^1(L^2, [0,T]) \cap C^0(H^2, [0,T])$$

via$(Q, u_0) \mapsto u$ is well-defined. To prove well-posedness, we shall fix $z = (Q, u_0) \in X$ and $z_1 = (Q_1, v_0) \in X$. Let $\|z\|_X = \max\{\|Q\|_{C^2[0,T]}, \|u_0\|_{2,2}\} < R$ and

$\|z_1\|_X < R$. The following notation is introduced:

$$w = \Delta u = v - u, \Delta z = z_1 - z = (\Delta Q, w_0) = (Q_1 - Q, v_0 - u_0)$$

Since $v = w + u$ satisfies (1.1), one has

$$i(w_t + u_t) = w_{xx} + u_{xx} + k|w + u|^2(w + u)$$

$$= w_{xx} + u_{xx} + k(w + u)^2(\bar{w} + \bar{u})$$

$$= w_{xx} + u_{xx} + k(|w|^2w + 2u|w|^2 + u^2\bar{w} + 2|u|^2w + \bar{u}w^2 + |u|^2u)$$

But u solves (1.1.1) thus the above equality can be simplified and w satisfies the following variable-coefficient, initial-value, boundary-value problem:

$$(1.3.1) \qquad iw_t = w_{xx} + k|w|^2w + 2ku|w|^2 + ku^2\bar{w} + 2k|u|^2w + k\bar{u}w^2$$

$$w(0,t) = \Delta Q, w_0 = v_0 - u_0.$$

LEMMA 1.3.1. *There exists $m > 0$ such that $\sup_{0\le t\le T} \|v - u\|_2 \le m\|z_1 - z\|^{\frac{1}{2}}_{X_0}$ where $X_0 = C[0,T] \times L^2[0,\infty)$.*

PROOF: We shall write $\Delta P = P_1 - P = v_x(0,t) - u_x(0,t)$. From (1.3.1) one has

$$(1.3.2) \qquad i\partial_t|w|^2 = iw_t\bar{w} + iw\bar{w}_t$$

$$= [w_{xx} + k|w|^2w + 2k|w|^2u + ku^2\bar{w} + 2kw|u|^2 + kw^2\bar{u}]\bar{w}$$

$$-w[\bar{w}_{xx} + k|w|^2\bar{w} + 2k|w|^2\bar{u} + k\bar{u}^2w + 2k\bar{w}|u|^2 + k\bar{w}^2u]$$

$$= w_{xx}\bar{w} - w\bar{w}_{xx} + 2k|w|^2(\bar{w}u - \bar{u}w)$$

$$+k(u^2\bar{w}^2 - \bar{u}^2w^2) + k|w|^2(\bar{u}w - u\bar{w})$$

$$= 2iIm(w_{xx}\bar{w} + 2k|w|^2\bar{w}u + ku^2\bar{w}^2 + k|w|^2\bar{u}w)$$

Thus

$$\int_0^\infty |w|^2 dx = \|w_0\|_2^2 \tag{1.3.3}$$

$$+2Im \int_0^t [\int_0^\infty (w_{xx}\bar{w} + 2k|w|^2\bar{w}u + ku^2\bar{w}^2 + k|w|^2\bar{u}w)dx]d\tau$$

$$= \|w_0\|_2^2 - 2Im \int_0^t \Delta P \Delta \bar{Q} d\tau + 2Im \int_0^t \int_0^\infty (2k|w|^2\bar{w}u + ku^2\bar{w}^2 + k|w|^2\bar{u}w)dxd\tau$$

$$\leq \|w_0\|_2^2 + 2(\int_0^t |\Delta Q(\tau)|^2 d\tau)^{\frac{1}{2}} (\int_0^t |\Delta P(\tau)|^2 d\tau)^{\frac{1}{2}}$$

$$+2|k| \int_0^t \int_0^\infty (3|w|^3|u| + |u|^2|w|^2) dxd\tau$$

By Corollary 1.2.2, for $0 \leq t \leq T, 0 \leq x < \infty$ one has

$$\sup |w(x,t)| \leq \sup(|u(x,t)| + |v(x,t)|) \leq 2\tilde{\lambda}\tilde{\lambda}_N \tag{1.3.4}$$

From (1.2.10) and Proposition 1.2.1,

$$(\int_0^t |\Delta P(\tau)|^2 d\tau)^{\frac{1}{2}} \leq [\int_0^t (2|P_1(\tau)|^2 + 2|P(\tau)|^2) d\tau]^{\frac{1}{2}} \tag{1.3.5}$$

$$\leq \sqrt{2}(\int_0^t |P_1(\tau)|^2 d\tau)^{\frac{1}{2}} + \sqrt{2}(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}}$$

$$\leq \sqrt{2}(2R\sqrt{T} + \sqrt{T\frac{|k|}{2}}R^2 + R + \frac{\|v'\|_2}{\sqrt{2}})$$

$$+\sqrt{2}(2R\sqrt{T} + \sqrt{T\frac{|k|}{2}}R^2 + R + \frac{\|u'\|_2}{\sqrt{2}})$$

$$\leq 4\sqrt{2T}R + 2\sqrt{T|k|}R^2 + 2\sqrt{2}R + 2\tilde{\lambda}_N = c_0$$

Putting (1.3.4),(1.3.5) in (1.3.3) and noting $\sup|u(x,t)| \leq \tilde{\lambda}\tilde{\lambda}_N$ one has

$$(1.3.6) \qquad \int_0^\infty |w|^2 dx \le \|w_0\|_2^2 + 2\sqrt{T}c_0\|\Delta Q\|_{C[0,T]}$$

$$+2|k|\int_0^t\int_0^\infty [3(2\tilde{\lambda}\tilde{\lambda}_N(R))|w|^2\tilde{\lambda}\tilde{\lambda}_N(R) + (\tilde{\lambda}\tilde{\lambda}_N(R))^2|w|^2]dxd\tau$$

$$\le \|w_0\|_2^2 + \hat{c}\|\Delta Q\|_{C[0,T]} + \tilde{c}\int_0^t\int_0^\infty |w|^2 dxd\tau$$

By Gronwall lemma,

$$(1.3.7) \qquad \int_0^\infty |w|^2 dx \le (\|w_0\|_2^2 + \hat{c}\|\Delta Q\|_{C[0,T]})e^{\tilde{c}t} \le (\|w_0\|_2^2 + \hat{c}\|\Delta Q\|_{C[0,T]})e^{\tilde{c}T}$$

Therefore

$$(1.3.8) \qquad \sup_{0\le t\le T}(\int_0^\infty |w|^2 dx)^{\frac{1}{2}} \le \sqrt{(\|w_0\|_2^2 + \hat{c}\|\Delta Q\|_{C[0,T]})e^{\tilde{c}T}}$$

$$\le m_0(\|w_0\|_2 + c\|\Delta Q\|_{C[0,T]}^{\frac{1}{2}}) = m_0(\|w_0\|_2^{\frac{1}{2}}\|w_0\|_2^{\frac{1}{2}} + c\|\Delta Q\|_{C[0,T]}^{\frac{1}{2}})$$

$$\le m_0(\sqrt{\|u_0\|_2 + \|v_0\|_2}\|w_0\|_2^{\frac{1}{2}} + c\|\Delta Q\|_{C[0,T]}^{\frac{1}{2}})$$

$$\le m_0(\sqrt{2R}\|w_0\|_2^{\frac{1}{2}} + c\|\Delta Q\|_{C[0,T]}^{\frac{1}{2}}) \le \max\{m\|w_0\|_2^{\frac{1}{2}}, m\|\Delta Q\|_{C[0,T]}^{\frac{1}{2}}\}$$

$$= m\|\Delta z\|_{X_0}^{\frac{1}{2}} = m\|z_1 - z\|_{X_0}^{\frac{1}{2}}$$

Our lemma is proved. Q.E.D.

LEMMA 1.3.2. *There exists* $\tilde{c} > 0$ *such that* $\|v - u\|_{Y_2} \le \tilde{c}\|z_1 - z\|_X^{\frac{1}{2}}$ *where* $Y_2 = C^1(L^2, [0,T])$.

PROOF: The norm of u on Y_2 is $\sup_{0\le t\le T}(\|u_t\|_2 + \|u\|_2)$. Since $\sup_{0\le t\le T}\|u\|_{2,2} < \infty$, it is clear that the RHS of (1.3.1) satisfies the local-Lipschitz condition on w in H^2. Hence one can adopt the proof of Theorem 4.1 in [18] to show that $w = v - u$ is the unique global classical solution satisfying (1.3.1).

We use change of variables via $w = W + \Delta Q(t)e^{-x}$ to rewrite (1.3.1) as follows

$$W_t = -iW_{xx} - ik|W|^2W + G_0 + G_1 + G_2 \tag{1.3.9}$$

$$G_0 = -\Delta Q'e^{-x} - i\Delta Qe^{-x} - 2ki|\Delta Q|^2e^{-2x}u - 2ki|u|^2\Delta Qe^{-x}$$
$$-ik\bar{u}\Delta^2Qe^{-2x} - kiu^2\Delta\bar{Q}e^{-x} - ki|\Delta Q|^2\Delta Qe^{-3x} \tag{1.3.10}$$

$$G_1 = -4kie^{-x}Re\bar{W}\Delta Q - 2ki|u|^2\bar{W} - 2ki\bar{u}\Delta Qe^{-x}W$$
$$-kiu^2\bar{W} - ki\Delta^2Qe^{-2x}\bar{W} - 2ki|\Delta Q|^2e^{-2x}W \tag{1.3.11}$$

$$G_2 = -2kiu|W|^2 - ki\bar{u}W^2 - 2ki\Delta Qe^{-x}|W|^2 - ki\Delta\bar{Q}e^{-x}W^2 \tag{1.3.12}$$

with $W(x,0) = w_0(x) - \Delta Q(0)e^{-x} = W_0(x), W_0(0) = w_0(0) - \Delta Q(0) = 0$. Thus $G_1 = \alpha W + \beta\bar{W}, G_2 = \gamma|W|^2 + \delta W^2$ and G_0 is independent of W. For $0 \leq t \leq T$ it is clear that $G_0, \alpha, \beta, \gamma, \delta$ all belong to $C^1(L^2[0,\infty))$. Generally $G_1, G_2 \in D(A)$ but $G_0 \notin D(A)$. But since $Q \in C^2, G_0'(t)$ is continuous. By [53], one has $\int_0^t N(t-s)G_0(s)ds \in D(A)$ where $A = -D_x^2$ with $D(A) = \{W, W_{xx} \in L^2[0,\infty); W(0) = 0\}$ and $N(t) = \exp\{At\}$ being a strongly continuous contraction semigroup in L^2. One then converts (1.3.9) to an intergral equation

$$W(t) = N(t)W_0 + \int_0^t N(t-s)G_0ds + \int_0^t (G_1 + G_2 - ik|W|^2W)ds \tag{1.3.13}$$
$$= N(t)W_0 + \int_0^t N(t-s)G(s)ds = N(t)W_0 + \int_0^t N(s)G(t-s)ds$$

here $\|W\|_\infty$ and $\|w\|_\infty$ are bounded because by Propositions 1.2.1 and 1.2.2 one has

$$\|w\|_{1,2} \le \|u\|_{1,2} + \|v\|_{1,2} \le 2\tilde{\lambda}_N(R)$$

$$\|w\|_\infty \le \|u\|_\infty + \|v\|_\infty \le 2\tilde{\lambda}\tilde{\lambda}_N(R)$$

hence $\|W\|_{1,2}$ and $\|W\|_\infty$ are bounded. Similar to (2.6) in [18] one obtains

$$\|k|W|^2W\|_{2,2} \le c_0\|W\|_\infty^2\|W\|_{2,2} \le \bar{c}\|W\|_{2,2} \tag{1.3.14}$$

Note

$$\|\Delta Q\|_{C^2[0,T]} \le \|Q_1\|_{C^2[0,T]} + \|Q\|_{C^2[0,T]} \le 2R$$

Since $W_0 \in D(A)$, one has $(N(t)W_0)_t = N(t)AW_0$. By (1.3.13)

$$W_t(t) = (N(t)W_0)_t + N(t)G(0) + \int_0^t N(t)G'(t-s)ds \tag{1.3.15}$$

$$= N(t)(-iD_x^2W_0) + N(t)G(0) + \int_0^t N(t-s)G'(s)ds$$

Here

$$G(0) = G_0(0) + G_1(0) + G_2(0) - ik|W_0|^2W_0 \tag{1.3.16}$$

By (1.3.10), (1.3.11) and (1.3.12)

$$\|G_0(0)\|_2 \le c_1\|\Delta Q\|_{C^1[0,T]} \tag{1.3.17}$$

$$\|G_1(0)\|_2 \le c_2\|W_0\|_2 \tag{1.3.18}$$

$$\|G_2(0)\|_2 \le c_3\|W_0\|_2 \tag{1.3.19}$$

Since $N(t)$ is a contraction semigroup on L^2 one has

$$\|N(t)(-iD_x^2W_0)\|_2 \le c_4\|W_0\|_{2,2} \tag{1.3.20}$$

Put (1.3.17), (1.3.18), (1.3.19) and (1.3.20) in (1.3.16):

$$\|G(0)\|_2 \le c_0(\|\Delta Q\|_{C^1[0,T]} + \|W_0\|_{2,2}) \tag{1.3.21}$$

Again from (1.3.9), (1.3.10), (1.3.11) and (1.3.12)

$$G'(t) = G_0'(t) + G_1'(t) + G_2'(t) - ik(2|W|^2W_t + W^2\bar{W}_t) \tag{1.3.22}$$

$$\|G_0'(t)\|_2 \le c_5\|\Delta Q\|_{C^2[0,T]} \tag{1.3.23}$$

$$\|G_1'(t)\|_2 \le c_6(\|W\|_2 + \|W_t\|_2) \tag{1.3.24}$$

$$\|G_2'(t)\|_2 \le c_7(\|W\|_2 + \|W_t\|_2) \tag{1.3.25}$$

Put (1.3.23), (1.3.24) and (1.3.25) in (1.3.22) (note $\|W\|_\infty$ is bounded):

$$\|G'(t)\|_2 \le c_8(\|\Delta Q\|_{C^2[0,T]} + \|W\|_2 + \|W_t\|_2) \tag{1.3.26}$$

Now put (1.3.20), (1.3.21) and (1.3.26) in (1.3.15) (using the fact $N(t)$ is a contraction semigroup):

$$\|W_t\|_2 \le \|N(t)(-iD_x^2W_0)\|_2 + \|G(0)\|_2 + \int_0^t \|N(t-s)G'(s)\|_2 ds \tag{1.3.27}$$

$$\leq c_4\|W_0\|_{2,2} + c_0(\|\Delta Q\|_{C^1[0,T]} + \|W_0\|_{2,2})$$

$$+ \int_0^t c_8(\|\Delta Q\|_{C^2[0,T]} + \|W\|_2 + \|W_t\|_2)ds$$

By Gronwall lemma,

(1.3.28) $$\|W_t\|_2 \leq (c_4\|W_0\|_{2,2} + c_0(\|\Delta Q\|_{C^1[0,T]} + \|W_0\|_{2,2})$$

$$+c_8(T\|\Delta Q\|_{C^2[0,T]} + \int_0^t \|W\|_2 ds)\exp\{c_8 T\}$$

$$\leq c'(\|\Delta Q\|_{C^2[0,T]} + \|W_0\|_{2,2}) + \bar{c}\int_0^t \|W\|_2 ds$$

Since $w = W + \Delta Q(t)e^{-x}, w_t = W_t + \Delta Q'(t)e^{-x}$, one has

(1.3.29) $$\|w_t\|_2 \leq \|W_t\|_2 + \|\Delta Q\|_{C^1[0,T]}$$

$$\leq c'(\|\Delta Q\|_{C^2[0,T]} + \|w_0\|_{2,2} + c_0'\|\Delta Q\|_{C^1[0,T]})$$

$$+\bar{c}(\int_0^t (\|w\|_2 + \|\Delta Q\|_{C^1[0,T]})ds) + \|\Delta Q\|_{C^1[0,T]}$$

$$\leq c(\|\Delta Q\|_{C^2[0,T]} + \|w_0\|_{2,2}) + \bar{c}\int_0^t \|w\|_2 ds$$

Now we can use Lemma 1.3.1.

(1.3.30) $$\|v-u\|_{Y_2} = \sup_{0\leq t\leq T}(\|w_t\|_2 + \|w\|_2)$$

$$\leq \sup_{0\leq t\leq T}(c(\|\Delta Q\|_{C^2[0,T]} + \|w_0\|_{2,2}) + \bar{c}\int_0^t m\|\Delta z\|_{X_1}^{\frac{1}{2}}ds + m\|\Delta z\|_{X_1}^{\frac{1}{2}})$$

$$\leq c(\|\Delta Q\|_{C^2[0,T]} + \|w_0\|_{2,2}) + \bar{c}Tm\|\Delta z\|_{X_1}^{\frac{1}{2}} + m\|\Delta z\|_{X_1}^{\frac{1}{2}} \leq \hat{c}\|\Delta z\|_{X_2}^{\frac{1}{2}}$$

and Lemma 1.3.2 is proved. Q.E.D.

THEOREM 1.3.3. *The map $f : X \to Y$ is continuous (thus (1.1.1) is well-posed).*

PROOF: By Lemma 1.3.2, it suffices to show that there exists $M > 0$ such that $\|v - u\|_{Y_3} \leq M\|z_1 - z\|_X^{\frac{1}{2}}$ where $Y_3 = C^0(H^2, [0,T])$. From (1.3.1)

$$(1.3.31)\quad \|w_{xx}\|_2 \leq \|w_t\|_2 + |k|(\|w^3\|_2 + 2\|uw^2\|_2 + \|u^2w\|_2 + 2\|u^2w\|_2 + \|uw^2\|)$$

$$\leq \|w\|_{Y_2} + |k|(\|w^3\|_2 + 3\|u^2w\|_2 + 3\|w^2u\|_2)$$

Put (1.3.8) and (1.3.29) in (1.3.30):

$$\|w_{xx}\|_2 \leq \hat{c}\|\Delta z\|_{X_2}^{\frac{1}{2}} + |k|(\|w\|_\infty^2\|w\|_2 + 3\|u\|_\infty^2\|w\|_2 + 3\|w\|_\infty\|u\|_\infty\|w\|_2)$$

$$\leq \hat{c}\|\Delta z\|_{X_2}^{\frac{1}{2}} + |k|((2c)^2\|w\|_2 + 3c^2\|w\|_2 + 3(2c)c\|w\|_2)$$

$$= \hat{c}\|\Delta z\|_{X_2}^{\frac{1}{2}} + m'\|w\|_2 \leq \hat{c}\|\Delta z\|_{X_2}^{\frac{1}{2}} + m'm\|\Delta z\|_{X_0}^{\frac{1}{2}} \leq \tilde{c}\|\Delta z\|_{X_2}^{\frac{1}{2}}$$

By (1.3.8) and (1.3.31),

$$(1.3.32)\quad \|v - u\|_{Y_3} = \sup_{0 \leq t \leq T}(\|w_{xx}\|_2 + \|w\|_2)$$

$$\leq \tilde{c}\|\Delta z\|_{X_2}^{\frac{1}{2}} + m\|\Delta z\|_{X_0}^{\frac{1}{2}} \leq M\|\Delta z\|_{X_2}^{\frac{1}{2}}$$

Hence (1.3.32) combined with Lemma 1.3.2 shows that $f : X \to Y = Y_2 \cap Y_3$ is continuous at z. The proof of well-posedness of (1.1.1) is completed. Q.E.D.

REMARK 1.3.4. *One can extend our results such that for the following problem*

$$(1.3.33)\quad iu_t = u_{xx} + k|u|^\sigma u$$

$$u(x,0) = u_0(x), u(0,t) = Q(t), u_0(0) = Q(0)$$

with $k \le 0$, $u_0 \in H^2, Q \in C^2, 0 \le \sigma < \infty$, there exists a unique global solution $u \in C^1(L^2) \cap C^0(H^2)$. Further, one can show that the above problem is well-posed. For $k > 0$, we are not able to obtain the similar results for $\sigma > 2$. This is because the term $\|u\|_{2\sigma}^{2\sigma}$ will appear in (1.2.2) and cannot be ignored when $k > 0$. Thus the powers for $\|u\|_2, \|u'\|_2$ in (1.2.4) will increase and (1.2.8), (1.2.9) and Proposition 1.2.1 do not hold.

Over the past three decades there have been many studies on evolution equations which for certain initial data possess solutions that do not exist for all time. (See [5],[52],[56] for general information and reference on nonexistence theorems proved by blow-up methods). The global existence and uniqueness result (cf. Theorem 4.1 [18]) indicates that under the assumption $u_0 \in H^2[0,\infty), Q \in C^2[0,\infty)$, $\|u'\|_2$ is bounded on any finite interval $[0,T]$. Nevertheless, one can study the situation where $Q \in C^2[0,T)$ for some $T > 0$ and try to find a condition on Q such that $\|u'\|_2$ (here u is the solution of the forced NLS (1.1.1)) blows up at finite time T. It can be shown that $|Q(t)| \to \infty$ will cause blow-up in most cases. In most situations, u will also blow up when $\|u'\|_2$ blows up.

REMARK 1.3.5. *If $\int_0^t |Q'(\tau)|^2 d\tau \le M$ then $|Q(t)| \le |Q(0)| + (T\int_0^t |Q'(\tau)|^2 d\tau)^{\frac{1}{2}}$ $\le |Q(0)| + \sqrt{TM}$ and $\int_0^t |Q(\tau)|^2 d\tau \le T(|Q(0)| + \sqrt{TM})^2$ thus Lemmas (3.2)(3.5) of [18] still hold and consequently Theorem (3.6) of [18] claim that $\|u'\|_2$ is bounded on $[0,T)$. (In fact, $\|u\|_{2,2}$ is bounded on $[0,T)$.) No blow-up will occur in this case. One interesting question is what is going to happen if $Q \in C^2[0,T), |Q(t)| \le M$ on $[0,T)$ but $\int_0^T |Q'(\tau)| d\tau = \infty$. A perfect example is $Q(t) = \sin\frac{1}{T-t}$.*

Chapter 2.

AN INITIAL-BOUNDARY VALUE PROBLEM FOR THE NONLINEAR SCHRÖDINGER EQUATION

§2.1 PHYSICAL IMPLICATIONS.

Many physically important nonlinear evolution equations in 1+1 (i.e. in one spatial and one temporal dimensions) have been found to possess exact solutions by the method of inverse scattering transform (IST). The initial value problems on the infinite interval $-\infty < x < \infty$ for decaying [2], periodic [59] and self-similar potentials [31] have been studied, to name a few. However, there are many open questions on extending the IST to solve initial-boundary value problems, sometimes called forced integrable systems [14,19,23,25,32]. Many efforts have been made since to solve such problems. For example, the existence, uniqueness and well- posedness of solution to the Korteweg-de Vries equation for $0 \leq x, t < \infty$ where $u(x,0)$ and $u(0,t)$ are given has been proven in [8,9]. Also the following forced nonlinear Schrödinger equation (NLS) has been considered:

$$iu_t = u_{xx} + k|u|^2 u \tag{2.1.1}$$

$$u(x,0) = u_0(x), u(0,t) = Q(t), u_0(0) = Q(0)$$

with k real. The main approach is to determine the time evolution of spectral data so that the problem can be solved by inverse scattering and the framework and technique extend to general AKNS systems in many cases (cf. [10,11,12,15,19,24]). As we have stated in Chapter 1, there exists a unique global classical solution for (2.1.1) and it is well-posed. There is, however, another type of half-line problem for the NLS:

$$iu_t = u_{xx} + k|u|^2 u \tag{2.1.2}$$

$$u(x,0) = u_0(x), u_x(0,t) + \alpha u(0,t) = R(t)$$

where α is real. Solving such a problem has important physical implications. (cf. [23]) For example, (2.1.2) arises in the propagation of optical solitons [33]. Also, NLS with an additional term u_x on the right-hand side and $\alpha \to \infty$ models water waves [40]. It has been shown that for (2.1.2), the solution $u(x,t)$ can be obtained by solving a linear integral equation uniquely defined in terms of appropriate scattering data satisfying a single nonlinear integrodifferential equation uniquely defined in terms of the boundary condition (cf. Fokas, e.g.[4,23,25,26]).

In this chapter, we obtain the global solution to (2.1.2) by using PDE method similar to [18]. In §2.2 we prove that there exists a unique classical local solution of (2.1.2). In §2.3 we establish a uniform bound on $|u(x,t)|$ for any fixed interval $t \in [0,T]$ to prove that the unique local solution obtained is in fact a global one.

In addition to (1.1.2),(1.1.3),(1.1.4) and (1.1.5), we write

$$Q(t) = u(0,t), P(t) = u_x(0,t), R(t) = P(t) + \alpha Q(t) \tag{2.1.3}$$

$$R_0 = \sup_{0 \le t \le T} (|R(t)| + |R'(t)|) \tag{2.1.4}$$

Further, we shall use the following Gagliardo-Nirenberg estimates (cf. [50] for details):

$$\|D^j u\|_p \le c\|D^m u\|_r^a \|u\|_q^{1-a} \tag{2.1.5}$$

where $\frac{1}{p} = \frac{j}{n} + a(\frac{1}{r} - \frac{m}{n}) + \frac{1-a}{q}$ and $\frac{j}{m} \le a \le 1$. The constant c depends on j, m, n, p, q, and the region $x \in [0,\infty)$, but we only need $n = 1$ here.

§2.2 LOCAL EXISTENCE-UNIQUENESS.

We shall study the half-line NLS (2.1.2) for real k, α with assumptions that $u_0(x), R(t)$ have appropriate smoothness and satisfy the necessary compatibility conditions to ensure the existence of solution at $x = t = 0$.

LEMMA 2.2.1. *Let $A = -iD_x^2 + ia$, $D(A) = \{v : v \in L^2, v_{xx} \in L^2, v'(0) + \alpha v(0) = 0\}$. Then the operator A is the infinitesimal generator of a continuous semigroup of contractors $N(t) = \exp At$ for $t \geq 0$. Here a is an appropriate positive constant depending on α.*

PROOF: The 'damping' term iav we added here is crucial to our proof in case $\alpha > 0$. Let $X = \{v : v \in L^2[0,\infty), v_{xx} \in L^2[0,\infty)\}$. Then X is a Banach space with a norm equivalent to $H^2[0,\infty)$-norm. Let $H = L^2[0,\infty), V = \{v \in H^1[0,\infty) : v'(0) + \alpha v(0) = 0\}$ then $D(A)$ and V are dense in H. Note $\|u\|_{0,\infty} \leq c\|u\|_{2,2}, \|u'\|_{0,\infty} \leq c'\|u\|_{2,2}$ by (2.1.5). To prove A is a closed operator, let $v_n \in D(A), Av_n \to y, v_n \to z$ in H. Then clearly $\{v_n\}$ is a Cauchy sequence in X. Completeness of X implies that $\{v_n\}$ converges in X and $Az = y$. Finally,

$$(2.2.1) \qquad |z'(0) + \alpha z(0)| \leq |v'(0) + \alpha v(0)| + |z'(0) - v'(0)| + |\alpha||z(0) - v(0)|$$

$$\leq c'\|z - v\|_X + c|\alpha|\|z - v\|_X \to 0$$

This shows that $z \in D(A)$. Thus A is closed. To show that the resolvent set of A contains R^+, let $v \in V$. Consider

$$(2.2.2) \qquad ((\lambda - A)v, v) = \int_0^\infty (\lambda v - iav + iv_{xx})\bar{v}dx$$

$$= (\lambda - ia)\int_0^\infty |v|^2 dx + i\int_0^\infty v_{xx}\bar{v}dx$$

$$= (\lambda - ia)\|v\|_2^2 + iv'\bar{v}\Big|_0^\infty - i\int_0^\infty |v'|^2 dx$$

$$= (\lambda - ia)\|v\|_2^2 - iv'(0)\bar{v}(0) - i\|v'\|_2^2 = (\lambda - ia)\|v\|_2^2 + i\alpha|v(0)|^2 - i\|v'\|_2^2$$

Take the imaginary part of (2.2.2),

$$|((\lambda - A)v, v)| \geq |a\|v\|_2^2 - \alpha|v(0)|^2 + \|v'\|_2^2| \tag{2.2.3}$$

By (2.1.5), there exists $c > 0$ such that $|v(0)| \leq \|v\|_\infty \leq c\|v'\|_2^{\frac{1}{2}}\|v\|_2^{\frac{1}{2}}$. Then the above inequality (2.2.3) becomes

$$|((\lambda - A)v, v)| \geq a\|v\|_2^2 - \alpha c^2\|v'\|_2\|v\|_2 + \|v'\|_2^2$$
$$\geq a\|v\|_2^2 - \frac{1}{2}\alpha^2 c^4\|v\|_2^2 - \frac{1}{2}\|v'\|_2^2 + \|v'\|_2^2 = (a - \frac{1}{2}\alpha^2 c^4)\|v\|_2^2 + \frac{1}{2}\|v'\|_2^2$$

If one sets $a > \frac{\alpha^2 c^4}{2}$ then

$$|((\lambda - A)v, v)| \geq c_0(\|v\|_2^2 + \|v'\|_2^2) = c_0\|v\|_V^2 \tag{2.2.4}$$

By Theorem 2.3.3 of [11], (2.2.4) implies that for $\lambda > 0$, the operator $\lambda - A$ maps $D(A)$ 1-1 onto H.

Now let $v \in D(A)$. For $\lambda > 0$, by taking the real part of (2.2.2), one has the following inequality $\lambda\|v\|_2 \leq \|(\lambda - A)v\|_2$ thus $\|(\lambda - A)^{-1}\| \leq \frac{1}{\lambda}$. By Hille-Yosida Theorem (§1.3.1 [53]), the unbounded and linear operator A is the infinitesimal generator of a continuous semigroup of a contractions $N(t) = \exp At$ for $t \geq 0$. Q.E.D.

In order to apply Lemma 2.2.1, we need to use the standard linear technique of change of variables. For α real, $u \in H^2[0, \infty), R(t) \in C^2[0, \infty)$, set up

$$v = u - S(t)e^{-bx}, b = |\alpha| + 1 > 0, S(t) = \frac{R(t)}{\alpha - b} \tag{2.2.5}$$

One has $v \in H^2[0, \infty)$ and $v_x = u_x + bS(t)e^{-bx}$. Thus

$$(2.2.6) \qquad v_x(0,t) + \alpha v(0,t) = u_x(0,t) + bS(t) + \alpha(u(0,t) - S(t))$$

$$= P(t) + bS(t) + \alpha Q(t) - \alpha S(t) = R(t) + (b - \alpha)S(t) = 0$$

and (1.1.2) becomes

$$(2.2.7) \qquad v_t = -iv_{xx} + iav - ik|v|^2 v + G_0 + G_1 + G_2$$

with $v_x(0,t) + \alpha v(0,t) = 0, v \in H^2[0,\infty)$ and

$$(2.2.8) \qquad G_0(x,t) = -i(k|S(t)|^2 S(t)e^{-3bx} - S'(t)e^{-bx})$$

$$(2.2.9) \qquad G_1(x,t,v) = -ik(S^2(t)e^{-2bx}\bar{v} + 2|S(t)|^2 e^{-2bx}v) - iav$$

$$(2.2.10) \qquad G_2(x,t,v) = -ik(\bar{S}(t)e^{-bx}v^2 + 2S(t)e^{-bx}|v|^2)$$

and $v \in D(A)$. Here one notes that $N(t)v_0, G_1, G_2 \in D(A)$ and generally $G_0 \notin D(A)$. Since $S(t) \in C^2[0,\infty), b > 0$, $G_0 = G_0(e^{-x}, S, S')$, $G_0'(t)$ is continuous. Thus by [53] one has $\int_0^t N(t-s)G_0(s)ds \in D(A)$. By Lemma 2.2.1, one can converts (2.2.7) to an integral equation:

$$(2.2.11) \quad v = N(t)v_0 + \int_0^t N(t-s)G_0(s)ds + \int_0^t N(t-s)(G_0 + G_1 - ik|v|^2 v)ds$$

By similar analysis as in [18], $H(x,t,v) = G_0 + G_1 + G_2 + G_3$ is locally Lipschitz in v under the norm of $D(A)$ uniformly on $[0,T]$ where $G_3(x,t,v) = -ik|v|^2 v$. Also for each $v \in D(A), H$ is continuous from $[0,T]$ into $D(A)$. Thus one can use Theorem 6.1.7 in [53] to obtain the following local existence-uniqueness theorem.

THEOREM 2.2.2 (LOCAL EXISTENCE-UNIQUENESS). *For $R(t) \in C^2, u_0(x) \in H^2$, there exists a unique classical solution v for equation (2.2.5) (hence u for equation (2.1.2)) such that $v, u \in C^1([0, T_M), D(A)) \cap L^2([0, T_M), D(A))$ with either $\lim \|u\|_{D(A)} = \infty$ as $t \to T_M$ or $T_M = \infty$.*

Next we shall prove that the local solution obtained here is indeed a global one.

§2.3 GLOBAL EXISTENCE.

The main objective in this section is to establish global existence for (2.1.2). Let u be a solution to the NLS. The following identities were established in [18] (recall that $Q(t) = u(0,t), P(t) = u_x(0,t), R(t) = P(t) + \alpha Q(t)$):

$$\|u\|_2^2 = \|u_0\|_2^2 - 2Im \int_0^t P(\tau)\overline{Q(\tau)}d\tau \tag{2.3.1}$$

$$\|u'\|_2^2 - \frac{k}{2}\|u\|_4^4 = \|u_0\|_2^2 - \frac{k}{2}\|u_0\|_4^4 - 2Re \int_0^t P(\tau)\overline{Q'(\tau)}d\tau \tag{2.3.2}$$

$$\int_0^\infty u\bar{u}'dx = \int_0^\infty u_0\bar{u}_0'dx - \int_0^t Q(\tau)\overline{Q'(\tau)}d\tau \tag{2.3.3}$$

$$+i\int_0^t |P(\tau)|^2 d\tau + i\frac{k}{2}\int_0^t |Q(\tau)|^4 d\tau$$

The boundary condition here is different from that in [18] but still we could use the above estimates and replace $P(t)$ by $R(t) - \alpha Q(t)$.

PROPOSITION 2.3.1. *For any $T > 0$, assume $R(t) \in C^1, u_0 \in H^1$ and the initial-boundary data satisfy necessary compatibility conditions at $x = t = 0$. Then $\|u\|_2$ and $\|u'\|_2$ are bounded on $[0, T]$.*

PROOF: Since $R(t) \in C^1$, there exists $R_0 > 0$ such that for $t \in [0,T], |R(t)| + |R'(t)| \leq R_0$. Now $u_0 \in H^1[0,\infty)$, by (2.1.5), there exists $\lambda > 0$ such that $\|u_0\|_4^4 \leq \lambda \|u_0'\|_2 \|u_0\|_2^3$. Take the real part of (2.3.3) and use the Cauchy-Schwartz inequality,

$$(2.3.4) \qquad Re \int_0^\infty u\bar{u}'dx = Re \int_0^\infty u_0\bar{u}_0'dx - Re \int_0^t Q(\tau)\overline{Q'(\tau)}d\tau$$

$$= Re \int_0^\infty u_0\bar{u}_0'dx - \frac{1}{2}(|Q(t)|^2 - |Q(0)|^2)$$

Thus

$$(2.3.5) \qquad |Q(t)|^2 = |Q(0)|^2 + Re \int_0^\infty u_0\bar{u}_0'dx - Re \int_0^\infty u\bar{u}'dx$$

$$= |u_0(0)|^2 + \|u_0'\|_2\|u_0\|_2 - Re \int_0^\infty u\bar{u}'dx \leq c_0 + \|u\|_2\|u'\|_2$$

From (2.3.1), (2.3.5)

$$(2.3.6) \qquad \|u\|_2^2 = \|u_0\|_2^2 - 2Im \int_0^t (R(\tau) - \alpha Q(\tau))\bar{Q}(\tau)d\tau$$

$$= \|u_0\|_2^2 - 2Im \int_0^t R(\tau)\bar{Q}(\tau)d\tau$$

$$\leq \|u_0\|_2^2 + 2(\int_0^t |R(\tau)|^2 d\tau)^{\frac{1}{2}}(\int_0^t |Q(\tau)|^2 d\tau)^{\frac{1}{2}}$$

$$\leq \|u_0\|_2^2 + 2(\int_0^t |R(\tau)|^2 d\tau)^{\frac{1}{2}}(\int_0^t (c_0 + \|u\|_2\|u'\|_2)d\tau)^{\frac{1}{2}}$$

$$\leq \|u_0\|_2^2 + 2(\int_0^t |R(\tau)|^2 d\tau)^{\frac{1}{2}}(c_0T)^{\frac{1}{2}} + 2(\int_0^t |R(\tau)|^2 d\tau)^{\frac{1}{2}}(\int_0^t \|u\|_2\|u'\|_2 d\tau)^{\frac{1}{2}}$$

$$\leq \|u_0\|_2^2 + 2\sqrt{T}R_0(c_0T)^{\frac{1}{2}} + 2\sqrt{t}R_0(\int_0^t \|u\|_2\|u'\|_2 d\tau)^{\frac{1}{2}}$$

$$= \hat{c} + 2\sqrt{t}R_0(\int_0^t \|u\|_2\|u'\|_2 d\tau)^{\frac{1}{2}}$$

From (2.3.2), (2.3.5)

$$\|u'\|_2^2 = \frac{k}{2}\|u\|_4^4 + \|u_0'\|_2^2 - \frac{k}{2}\|u_0\|_4^4 - 2Re\int_0^t P(\tau)\overline{Q'(\tau)}d\tau \tag{2.3.7}$$

$$= \frac{k}{2}\|u\|_4^4 + c - 2ReP(t)\bar{Q}(t) + 2ReP(0)\bar{Q}(0) + 2Re\int_0^t P'(\tau)\bar{Q}(\tau)d\tau$$

$$= \frac{k}{2}\|u\|_4^4 + c - 2Re(R(t) - \alpha Q(t))\bar{Q}(t) + 2Re(R(0) - \alpha Q(0))\bar{Q}(0)$$

$$+2Re\int_0^t (R'(\tau) - \alpha Q'(\tau))\bar{Q}(\tau)d\tau$$

$$= \frac{k}{2}\|u\|_4^4 + c - 2ReR(t)\bar{Q}(t) + 2\alpha|Q(t)|^2 + 2Re(R(0) - \alpha u_0(0))\bar{u}_0(0)$$

$$+2Re\int_0^t R'(\tau)\bar{Q}(\tau)d\tau - 2\alpha Re\int_0^t Q(\tau)\overline{Q'(\tau)}d\tau$$

$$= \frac{k}{2}\|u\|_4^4 + c - 2ReR(t)\bar{Q}(t) + 2\alpha|Q(t)|^2 + 2ReR(0)\bar{u}_0(0) - 2\alpha|u_0(0)|^2$$

$$+2Re\int_0^t R'(\tau)\bar{Q}(\tau)d\tau - \alpha|Q(t)|^2 + \alpha|Q(0)|^2$$

$$= \frac{k}{2}\|u\|_4^4 + c - 2ReR(t)\bar{Q}(t) + \alpha|Q(t)|^2 + 2ReR(0)\bar{u}_0(0)$$

$$-\alpha|u_0(0)|^2 + 2Re\int_0^t R'(\tau)\bar{Q}(\tau)d\tau$$

$$\leq \frac{|k|}{2}\|u\|_4^4 + c' + |\alpha||Q(t)|^2 + 2R_0|Q(t)| + 2\int_0^t R_0|Q(\tau)|d\tau$$

$$\leq \frac{|k|}{2}\|u\|_4^4 + c' + |\alpha||Q(t)|^2 + R_0^2 + |Q(t)|^2 + \int_0^t (R_0^2 + |Q(\tau)|^2)d\tau$$

$$\leq \frac{|k|}{2}\|u\|_4^4 + c' + (|\alpha| + 1)|Q(t)|^2 + R_0^2 + R_0^2 T + \int_0^t |Q(\tau)|^2 d\tau$$

By (2.1.5) there exists $\lambda > 0$ such that $\|u\|_4^4 \leq \lambda\|u'\|_2\|u\|_2^3$. Thus (2.3.7) becomes (using (2.3.5), $0 \leq t \leq T$)

$$(2.3.8) \qquad \|u'\|_2^2 \leq \frac{|k|}{2}\lambda\|u'\|_2\|u\|_2^3 + c' + (|\alpha|+1)(c_0 + \|u\|_2\|u'\|_2)$$

$$+R_0^2(1+T) + \int_0^t (c_0 + \|u\|_2\|u'\|_2)d\tau$$

$$\leq \bar{c} + \frac{|k|}{2}\lambda\|u'\|_2\|u\|_2^3 + (|\alpha|+1)\|u\|_2\|u'\|_2 + \int_0^t \|u\|_2\|u'\|_2 d\tau$$

By (2.3.6) one has

$$(2.3.9) \qquad \|u\|_2^2 \leq \hat{c} + 2\sqrt{t}R_0(\int_0^t \|u\|_2\|u'\|_2 d\tau)^{\frac{1}{2}}$$

$$\leq \max\{2\hat{c}, 4\sqrt{t}R_0(\int_0^t \|u\|_2\|u'\|_2 d\tau)^{\frac{1}{2}}\}$$

Let

$$(2.3.10) \qquad f(t) = \sup_{0\leq\tau\leq t} \|u\|_2, g(t) = \sup_{0\leq\tau\leq t} \|u'\|_2$$

Then (2.3.9) implies

$$(2.3.11) \qquad f^2(t) = \sup_{0\leq\tau\leq t} \|u\|_2^2 \leq \sup_{0\leq\tau\leq t} \max\{2\hat{c} + 2\sqrt{\tau}R_0(\int_0^\tau \|u\|_2\|u'\|_2 d\tau)^{\frac{1}{2}}\}$$

$$\leq \max\{2\hat{c}, 4\sqrt{t}R_0(\int_0^t \|u\|_2\|u'\|_2 d\tau)^{\frac{1}{2}}\}$$

$$\leq \max\{2\hat{c}, 4tR_0(f(t)g(t))^{\frac{1}{2}}\}$$

If $2\hat{c} \geq 4tR_0(f(t)g(t))^{\frac{1}{2}}$ then (2.3.11) implies

$$(2.3.12) \qquad f^3(t) = (f^2(t))^{\frac{3}{2}} \leq (2\hat{c})^{\frac{3}{2}}$$

Otherwise

$$(2.3.13) \qquad f^2(t) \leq 4R_0t(f(t)g(t))^{\frac{1}{2}}$$

which implies that

$$f^3(t) \leq 16R_0^2t^2g(t) \tag{2.3.14}$$

In any event, by combining (2.3.12) and (2.3.14) one obtains

$$f^3(t) \leq \max\{(\hat{c})^{\frac{3}{2}}, 16R_0t^2g(t)\} \leq (2\hat{c})^{\frac{3}{2}} + 16R_0^2t^2g(t) \tag{2.3.15}$$

(2.3.12) and (2.3.14) also imply that

$$f^2(t) \leq 2\hat{c} + (16R_0^2t^2g(t))^{\frac{2}{3}} \tag{2.3.16}$$

By (2.3.8) and (2.3.15)

$$\begin{aligned}
(2.3.17)\quad g^2(t) &\leq \sup_{0\leq\tau\leq t}[\bar{c} + \frac{|k|}{2}\lambda\|u'\|_2\|u\|_2^3 + (|\alpha|+1)\|u\|_2\|u'\|_2 + \int_0^\tau \|u\|_2\|u'\|_2 d\tau] \\
&\leq \bar{c} + \frac{|k|}{2}\lambda g(t)f^3(t) + (|\alpha|+1)f(t)g(t) + \int_0^t f(\tau)g(\tau)d\tau \\
&\leq \bar{c} + \frac{|k|}{2}\lambda g(t)((2\hat{c})^{\frac{3}{2}} + 16R_0^2t^2g(t)) \\
&+\frac{1}{2}(2(|\alpha|+1)^2f^2(t)) + \frac{1}{2}(\frac{g^2(t)}{2}) + \int_0^t f(\tau)g(\tau)d\tau \\
&\leq \bar{c} + \frac{1}{2}(k\lambda(2\hat{c})^{\frac{3}{2}})^2 + \frac{1}{2}g^2(t) + \frac{|k|}{2}\lambda R_0^2g^2(t)16t^2 \\
&+(|\alpha|+1)^2f^2(t) + \frac{1}{4}g^2(t) + \int_0^t f(\tau)g(\tau)d\tau \\
&= \frac{\tilde{c}}{4} + \frac{3}{4}g^2(t) + 8|k|\lambda R_0^2g^2(t)t^2 + (|\alpha|+1)^2f^2(t) + \int_0^t f(\tau)g(\tau)d\tau
\end{aligned}$$

Hence

$$g^2(t) \leq \tilde{c} + 32|k|\lambda R_0^2t^2g^2(t) + 4(|\alpha|+1)^2f^2(t) + 4\int_0^t g(\tau)f(\tau)d\tau \tag{2.3.18}$$

Let $0 = t_0 < t_1 < t_2 ...... < t_n = T$ such that

(2.3.19) $$t_{i+1} - t_i < \frac{1}{\sqrt{64|k|\lambda R_0^2}}, i = 0, 1, 2, ..., n-1$$

Consider $0 \leq t \leq t_1$. Then (2.3.18) becomes (using (2.3.16))

$$g^2(t) \leq \tilde{c} + \frac{1}{2}g^2(t) + 4(|\alpha|+1)^2 f^2(t) + 4\int_0^t g(\tau)f(\tau)d\tau$$

$$\leq \tilde{c} + \frac{1}{2}g^2(t) + 4(|\alpha|+1)^2(2\hat{c} + (16R_0^2 t^2 g(t))^{\frac{2}{3}} + 4\int_0^t g(\tau)f(\tau)d\tau$$

$$\leq \tilde{c}' + \frac{1}{2}g^2(t) + \tilde{c}_0(g(t))^{\frac{2}{3}} + 4\int_0^t g(\tau)f(\tau)d\tau$$

If $g(t) > 1$ then $(g(t))^{\frac{2}{3}} < g(t)$. If $g(t) \leq 1$ then $(g(t))^{\frac{2}{3}} \leq 1$. In any event, $(g(t))^{\frac{2}{3}} \leq g(t) + 1$. Thus

(2.3.20) $$g^2(t) \leq \tilde{c}' + \frac{1}{2}g^2(t) + \tilde{c}_0(g(t)+1) + 4\int_0^t g(\tau)f(\tau)d\tau$$

$$\leq \tilde{c}' + \tilde{c}_0 + \frac{1}{2}g^2(t) + \frac{1}{2}(2\tilde{c}_0^2) + \frac{1}{2}(\frac{1}{2}g^2(t)) + 4\int_0^t g(\tau)f(\tau)d\tau$$

Therefore

(2.3.21) $$g^2(t) \leq 4(\tilde{c}' + \tilde{c}_0 + \frac{1}{2}\tilde{c}_0^2) + 16\int_0^t g(\tau)f(\tau)d\tau = c + 16\int_0^t g(\tau)f(\tau)d\tau$$

By (2.3.9)

(2.3.22) $$f^2(t) \leq \sup_{0 \leq \tau \leq t}(\hat{c} + 2\sqrt{t}R_0(\int_0^\tau \|u\|_2\|u'\|_2 d\tau)^{\frac{1}{2}}$$

$$\leq \hat{c} + 2\sqrt{T}R_0(\int_0^t \|u\|_2\|u'\|_2 d\tau)^{\frac{1}{2}} \leq \hat{c} + 2\sqrt{T}R_0(\int_0^t f(\tau)g(\tau)d\tau)^{\frac{1}{2}}$$

$$\leq \hat{c} + TR_0^2 + \int_0^t f(\tau)g(\tau)d\tau$$

Add (2.3.21), (2.3.22) together

$$\|u\|_{1,2}^2 \le \sup_{0\le\tau\le t} \|u(,\tau)\|_2^2 + \sup_{0\le\tau\le t} \|u_x(,\tau)\|_2^2 \le f^2(t)+g^2(t) \tag{2.3.23}$$

$$\le \hat{c} + TR_0^2 + \int_0^t f(\tau)g(\tau)d\tau + c + 16\int_0^t g(\tau)f(\tau)d\tau$$

$$\le c' + \frac{17}{2}\int_0^t (f^2(\tau)+g^2(\tau))d\tau$$

By Gronwall lemma,

$$\|u\|_{1,2}^2 \le f^2(t)+g^2(t) \le c'e^{8.5t} \le c'e^{8.5T} \le M \tag{2.3.24}$$

for $0 \le t \le t_1 < T$. One could repeat the above process for $t_1 \le t \le t_2$ to conclude that $\|u\|_{1,2}^2$ is bounded on $[t_1, t_2]$. By induction, $\|u\|_{1,2}^2$ is bounded on $[t_i, t_{i+1}]$ for $i = 1, 2, \ldots\ldots, n-1$, i.e. $\|u\|_2^2 \le M_0, \|u'\|_2^2 \le M_0$ on $[0,T]$. Thus $\|u\|_\infty$ is bounded on $[0,T]$ via (2.1.5) by setting $j=0, p=\infty, r=2, m=2, q=2, a=\frac{1}{2}$:

$$\|u\|_\infty \le \tilde{\lambda}\|u\|_2^{\frac{1}{2}}\|u'\|_2^{\frac{1}{2}} \le \tilde{\lambda}\sqrt{M_0} \tag{2.3.25}$$

and our proof is now complete. Q.E.D.

THEOREM 2.3.2 (GLOBAL EXISTENCE). *Under the assumptions of Theorem 2.2.2, the local solution u of NLS (2.1.2) is a global solution, i.e. $T_M = \infty$.*

PROOF: By Theorem 2.2.2, it suffices to show that for any $T > 0$, $\|u\|_{2,2}$ is bounded on $[0,T]$. Since $v = u - S(t)e^{-bx}$, $\|u\|_{1,2} \le \sqrt{M}$, one has also $\|v\|_p \le c$ and $\|v'\|_2 \le c$ for $p \ge 2$ by (2.1.5). From (2.2.9) and the definition of $H(x,t,v) = G_0 + G_1 + G_2 + G_3$ one has

$$v_t(t) = (N(t)v_0)_t + N(t)H(x,0,v_0) + \int_0^t N(t)H_s(x,t-s,v)ds \tag{2.3.26}$$

$$= N(t)Av_0 + N(t)H(x,0,v_0) + \int_0^t N(t-s)H_s(x,s,v)ds$$

Since $N(t)$ is a contraction semigroup on L^2,

$$\|v_t(t)\|_2 \le \|Av_0\|_2 + \|H(x,0,v_0)\|_2 + \int_0^t \|H_s(x,s,v)ds\|_2 ds \tag{2.3.27}$$

$$\le c_0 + \int_0^t \|G_0'(s) + G_1'(s) + G_2'(s) + G_3'(s)\|_2 ds$$

Since $S(t) \in C^2[0,\infty)$, (2.2.6) implies (for $t \in [0,T]$) that

$$\|G_0'(t)\|_2 \le c_1 \tag{2.3.28}$$

By (2.2.7), $G_1'(t) = h(S, S', e^{-bx}, v, v_t)$. Since $\|v\|_2$ is bounded and $S \in C^2$, one has

$$\|G_1'(t)\|_2 \le c_2 + c_3\|v_t(t)\|_2 \tag{2.3.29}$$

By the same reason, one readily obtains

$$\|G_n'(t)\|_2 \le c_4 + c_5\|v_t(t)\|_2 \tag{2.3.30}$$

for $n = 2, 3$. Thus (2.3.27) becomes

$$\|v_t(t)\|_2 \le c' + \int_0^t (\tilde{c} + \hat{c}\|v_s(s)\|_2)ds \le c' + \tilde{c}T + \hat{c}\int_0^t \|v_s(s)\|_2 ds \tag{2.3.31}$$

on $[0,T]$. By Gronwall lemma one concludes $\|v_t(t)\|_2 \le \bar{c}$. Now by (2.2.5) one has

$$\|v_{xx}\|_2 \le \|v_t(t)\|_2 + \|iav + G_0 + G_1 + G_2 + G_3\|_2 \le c \tag{2.3.32}$$

because $\|v\|_p$ is bounded. Thus $\|v\|_{2,2}$ is bounded on $[0, T]$ for any T. Hence v is the global solution to (2.2.5) and u is the global solution to (2.1.2) via $v = u - S(t)e^{-bx}$. Q.E.D.

The PDE method to show the global existence of (2.1.2) is an approach different from the IST method in [23]. Along with [18], we try to provide some answers to the questions raised in [32]. We expect well-posedness of (2.1.2) will hold.

Chapter 3.
THE FULL-LINE PROBLEM FOR THE GINZBURG-LANDAU EQUATION (GL)

§3.1 ON THE GINZBURG-LANDAU EQUATION.

The Ginzburg-Landau equation (GL)

$$u_t = (\nu + i\alpha)u_{xx} - (\kappa + i\beta)|u|^2 u + \gamma u \tag{3.1.1}$$

is a modulation equation describing the nonlinear development of unstable waves in many physical systems (such as hydrodynamics) or chemical systems in which some kind of turbulence appears. (cf. [7,20,21,30,34,35,39,43-47,58]). It was originally derived by A. Newell in [42] and used to study parallel flow stability problem [29,55]. By studying the long-time behavior of solutions to the GL equation it was shown that a finite-dimensional attractor captures all the solutions (cf. [27]). The GL equation on periodic domain was studied to investigate possible soft and hard turbulence (cf.[6]) where a series of interesting estimates were given. The Cauchy problem for the GL equation with bounded domain and zero boundary condition or zero normal derivative boundary condition is well-posed by using classical techniques of nonlinear parabolic equations (cf.[38]). On the other hand, the initial-boundary value problem for nonlinear Schödinger equation (which in some sense, is a special Ginzburg-Landau equation) was examined in [18] and Chapter 1 where global existence-uniqueness and well-posedness were established. Though the NLS looks like a special GL equation, they are quite different. Because of the appearance of the dissipative term, the Ginzburg-Landau equation does not possess Hamiltonian structure and conserved quantities. Hence it is no longer an intergrable system like NLS. In fact, our results on the GL equation in this chapter and next chapter do not cover NLS. We shall, first of all, study the Cauchy problem for the following GL equation (so-called full-line problem):

(3.1.2) $$u_t = (\nu + i\alpha)u_{xx} - (\kappa + i\beta)|u|^2 u + \gamma u, \nu > 0, \kappa > 0$$

$$-\infty < x < \infty, 0 \le t < \infty$$

with $u(x, 0) = u_0(x) \in H^2(-\infty, \infty)$. In §3.2 we shall extend global existence theorem of the Cauchy problem for the GL equation in bounded domain to unbounded domain. The following notations are used throughout this chapter:

(3.1.3) $$\|u'\|_2 = [\int_{-\infty}^{\infty} |u_x(x,t)|^2 dx]^{\frac{1}{2}}$$

(3.1.4) $$\|u\|_p = [\int_{-\infty}^{\infty} |u(x,t)|^p dx]^{\frac{1}{p}}$$

(3.1.5) $$\|u\|_{m,p} = [\int_{-\infty}^{\infty} \sum_{|q| \le m} |D_x^q u(x,t)|^p dx]^{\frac{1}{p}}$$

§3.2 GLOBAL SOLUTIONS.

We shall study the following initial-value problem for the GL equation:

(3.2.1) $$u_t = (\nu + i\alpha)u_{xx} - (\kappa + i\beta)|u|^2 u + \gamma u$$

with $\nu > 0, \kappa > 0, u(x, 0) = u_0(x) \in H^2(\Omega)$. For results in case Ω is a finite domain with zero boundary condition see [27]. Now we shall assume $\Omega = (-\infty, \infty)$. First let us check $A = (\nu + i\alpha)D_x^2$ is a semi-group generator with $D(A) = F = \{u : u \in L^2, u_{xx} \in L^2\}$. For $u \in D(A), f \in L^2$, one has

$$(3.2.2) \qquad ((\lambda - A)u, u) = \int_{-\infty}^{\infty} (\lambda u - (\nu + i\alpha)u_{xx})\bar{u}dx$$

$$= \lambda\|u\|_2^2 - (\nu + i\alpha)u_x\bar{u}\,\Big|_{-\infty}^{\infty} + (\nu + i\alpha)\int_{-\infty}^{\infty} u_x\bar{u}_x dx$$

$$= \lambda\|u\|_2^2 + (\nu + i\alpha)\|u'\|_2^2$$

Let us check $\lambda - A$ is 1-1 and onto for $\lambda > 0$. From (3.2.2) one has

$$(*) \qquad |((\lambda - A)u, u)| \geq \lambda\|u\|_2^2 + \nu\|u'\|_2^2 \geq \min\{\lambda, \nu\}\|u\|_{H^1}^2$$

By Theorem 2.3.3 of [11], (*) implies that for $\lambda > 0$, $\lambda - A$ maps $D(A)$ 1-1 onto L^2 and the resolvent set of A contains R^+. (See also Theorem 1.9.2 of [12].) Now take the real part of (3.2.2)

$$(3.2.3) \qquad Re((\lambda - A)u, u) = \lambda\|u\|_2^2 + \nu\|u'\|_2^2 \geq \lambda\|u\|_2^2$$

Thus $\|(\lambda - A)^{-1}\| \leq \frac{1}{\lambda}$. By Hille-Yosida, A generates a strongly continuous contraction semigroup (cf.[53]) $N(t) = \exp\{At\}$ and one can write

$$(3.2.4) \qquad u(t) = N(t)u_0 + \int_0^t N(t-s)(-(\kappa + i\beta)|u|^2u + \gamma u)ds$$

For $H(u) = -(\kappa + i\beta)|u|^2u + \gamma u$ and $u, v \in F$, one has (similar to (2.7) in [18])

$$\|H(u) - H(v)\|_{2,2} \leq c(\|u\|_{2,2} + \|v\|_{2,2})\|u - v\|_{2,2} + |\gamma|\|u - v\|_{2,2}$$

Thus local Lipschitz condition on H is satisfied. Thus the basic theorems in §§6.1 and §§8.1 of [53] can be adopted here. One has

THEOREM 3.2.1. *Given $u_0(x) \in H^2(-\infty,\infty)$, (3.2.1) with initial data $u(x,0) = u_0(x) \in H^2(-\infty,\infty)$ has a unique classical solution $u \in C^1(L^2(-\infty,\infty)) \cap C^0(H^2(-\infty,\infty))$ on $[0, T_M)$ with either $T_M = \infty$ or $\lim_{t\to T_M^-} \|u\|_F = \infty$.*

To prove the global existence, we need some estimates on $\|u\|_2, \|u'\|_2$. First note

$$\partial_t \int_{-\infty}^{\infty} |u|^2 dx = \int_{-\infty}^{\infty} (u_t\bar{u} + u\bar{u}_t)dx \tag{3.2.5}$$

$$= \int_{-\infty}^{\infty} ((\nu + i\alpha)u_{xx}\bar{u} - (\kappa + i\beta)|u|^4 + \gamma|u|^2)dx$$

$$+ \int_{-\infty}^{\infty} ((\nu - i\alpha)\bar{u}_{xx}u - (\kappa - i\beta)|u|^4 + \gamma|u|^2)dx$$

$$= (\nu + i\alpha)u_x\bar{u}\Big|_{-\infty}^{\infty} + (\nu - i\alpha)\bar{u}_x u\Big|_{-\infty}^{\infty} - 2\kappa\|u\|_4^4$$

$$+2\gamma\|u\|_2^2 - (\nu + i\alpha)\|u'\|_2^2 - (\nu - i\alpha)\|u'\|_2^2$$

$$= -2\kappa\|u\|_4^4 + 2\gamma\|u\|_2^2 - 2\nu\|u'\|_2^2$$

Thus (3.2.5) implies that $\partial_t\|u\|_2^2 \leq 2\gamma\|u\|_2^2$ and $\|u\|_2^2 \leq \|u_0\|_2^2 + 2\gamma\int_0^t \|u\|_2^2 d\tau$. By Gronwall lemma,

$$\|u\|_2^2 \leq \|u_0\|_2^2 e^{2|\gamma|t} \leq M \tag{3.2.6}$$

for $0 \leq t \leq T$. Now one uses integration by parts to get

$$\partial_t \int_{-\infty}^{\infty} |u'|^2 dx = \int_{-\infty}^{\infty} (u_{xt}\bar{u}_x + u_x\bar{u}_{xt})dx - \int_{-\infty}^{\infty} (u_t\bar{u}_{xx} + u_{xx}\bar{u}_t)dx \tag{3.2.7}$$

$$= -\int_{-\infty}^{\infty} ((\nu + i\alpha)|u_{xx}|^2 - (\kappa + i\beta)|u|^2 u\bar{u}_{xx} + \gamma u\bar{u}_{xx})dx$$

$$-\int_{-\infty}^{\infty} ((\nu - i\alpha)|u_{xx}|^2 - (\kappa - i\beta)|u|^2\bar{u}u_{xx} + \gamma\bar{u}u_{xx})dx$$

$$= -2\nu\|u_{xx}\|_2^2 + 2\kappa Re\int_{-\infty}^{\infty} |u|^2 u\bar{u}_{xx}dx - 2\beta Im\int_{-\infty}^{\infty} |u|^2 u\bar{u}_{xx}dx$$

$$-\gamma u\bar{u}_x \Big|_{-\infty}^{\infty} + \gamma\int_{-\infty}^{\infty} |u_x|^2 dx - \gamma\bar{u}u_x \Big|_{-\infty}^{\infty} + \gamma\int_{-\infty}^{\infty} |u_x|^2 dx$$

$$= -2\nu\|u_{xx}\|_2^2 + 2\gamma\|u'\|_2^2 + 2\kappa Re\int_{-\infty}^{\infty} u\bar{u}_{xx}dx - 2\beta Im\int_{-\infty}^{\infty} |u|^2 u\bar{u}_{xx}dx$$

Let $m = 2\kappa + 2|\beta|, \delta = \frac{2\nu}{m}$. Then (3.2.7) becomes

$$\partial_t\|u'\|_2^2 \le -2\nu\|u_{xx}\|_2^2 + 2\gamma\|u'\|_2^2 + m\int_{-\infty}^{\infty} |u|^3|\bar{u}_{xx}|dx \tag{3.2.8}$$

$$\le -2\nu\|u_{xx}\|_2^2 + 2\gamma\|u'\|_2^2 + m\int_{-\infty}^{\infty} (\frac{|u|^6}{2\delta} + \frac{|u_{xx}|^2}{2}\cdot\delta)dx$$

$$\le -2\nu\|u_{xx}\|_2^2 + 2\gamma\|u'\|_2^2 + \frac{m^2}{4\nu}\|u\|_6^6 + \nu\|u_{xx}\|_2^2$$

$$= -\nu\|u_{xx}\|_2^2 + 2\gamma\|u'\|_2^2 + \frac{m^2}{4\nu}\|u\|_6^6$$

Thus

$$\|u'\|_2^2 \le \|u_0'\|_2^2 - \nu\int_0^t \|u_{xx}\|_2^2 d\tau + 2\gamma\int_0^t \|u'\|_2^2 d\tau + \frac{m^2}{4\nu}\int_0^t \|u\|_6^6 d\tau \tag{3.2.9}$$

By Gagliardo-Nirenberg estimates (cf. [50]), there exists $\lambda > 0$ such that

$$\|u\|_6^6 \le \lambda\|u'\|_2^2\|u\|_2^4 \tag{3.2.10}$$

for all u. Use (3.2.6) to obtain

$$\|u\|_6^6 \le \lambda\|u'\|_2^2\|u\|_2^4 \le \lambda\|u'\|_2^2 M^2 \tag{3.2.11}$$

Put (3.2.11) into (3.2.9) one has

$$\|u'\|_2^2 \le \|u_0'\|_2^2 - \nu\int_0^t \|u_{xx}\|_2^2 d\tau + 2|\gamma|\int_0^t \|u'\|_2^2 d\tau + \frac{m^2}{4\nu}\int_0^t \lambda\|u'\|_2^2 M^2 d\tau \tag{3.2.12}$$

$$\leq \|u_0'\|_2^2 + \int_0^t (2|\gamma| + \frac{m^2}{4\nu}\lambda M^2)\|u'\|_2^2 d\tau = \|u_0'\|_2^2 + c_0 \int_0^t \|u'\|_2^2 d\tau$$

By Gronwall lemma, for $0 \leq t \leq T$ there exists $C_0 > 0$ such that

$$\|u'\|_2^2 \leq \|u_0'\|_2^2 e^{c_0 t} \leq \|u_0'\|_2^2 e^{c_0 T} \leq C_0 \tag{3.2.13}$$

Thus by estimates in [50], one uses (2.2.6) and (2.2.13) to get

$$\|u\|_\infty \leq \lambda_0 \|u\|_2^{\frac{1}{2}} \|u'\|_2^{\frac{1}{2}} \leq \lambda_0 (MC_0)^{\frac{1}{4}} = C \tag{3.2.14}$$

Since N is also a contraction operator in $F = D(A)$, by (3.2.4) one has

$$\|u(t)\|_F \leq \|N(t)u_0\|_F + \int_0^t \|N(t-s)(-(\kappa + i\beta)|u|^2 u + \gamma u)\|_F ds \tag{3.2.15}$$

$$\leq \|u_0\|_F + \int_0^t (\tilde{c}\|u^3\|_F + c'\|u\|_F) ds$$

Since F-norm is equivalent to H^2-norm, one has (using (3.2.14))

$$\|u(t)\|_{2,2} \leq \bar{c}\|u\|_F \leq \bar{c}\|u_0\|_F + \int_0^t (\tilde{c}\|u^3\|_F + c'\|u\|_F) ds \tag{3.2.16}$$

$$\leq \hat{c} + m' \int_0^t (\tilde{c}\|u^3\|_{2,2} + c'\|u\|_{2,2}) ds \leq \hat{c} + m' \int_0^t (\tilde{c}\|u\|_\infty^2 \|u\|_{2,2} + c'\|u\|_{2,2}) ds$$

$$\leq \hat{c} + m' \int_0^t (\tilde{c}C^2 \|u\|_{2,2} + c'\|u\|_{2,2}) ds \leq \hat{c} + \tilde{m} \int_0^t \|u\|_{2,2} ds$$

By Gronwall lemma again one concludes that for $0 \leq t \leq T$

$$\|u\|_{2,2} \leq \hat{c} e^{\tilde{m}t} \leq \hat{c} e^{\tilde{m}T} < \infty \tag{3.2.17}$$

Thus for any $T > 0$, $\|u\|_{2,2} < \infty$ for $0 \leq t \leq T$. Therefore the solution u of (3.2.1) is global solution and we have completed the proof of the following result:

THEOREM 3.2.2. *For the full-line Ginzburg-Landau problem (3.2.1), there exists a unique global classical solution, i.e.* $T_M = \infty$ *in Theorem 3.2.1.*

COROLLARY 3.2.3. *Assume* $\gamma < 0$, *then* $\|u\|_2^2 \leq \|u_0\|_2^2$ *for all* t. *Further, if* $\gamma < -\frac{m^2}{16\nu}\lambda\|u_0\|_2^4$ *where* $m = 2\kappa+2|\beta|$ *then* $\|u'\|_2^2 \leq \|u_0'\|_2^2$ *thus* $|u(x,t)|$ *is uniformly bounded on the whole quarter plane* $0 \leq t < \infty, 0 \leq x < \infty$.

PROOF: From (3.2.5) one has $\partial_t\|u\|_2^2 \leq 2\gamma\|u\|_2^2 \leq 0$ thus $\|u\|_2^2 \leq \|u_0\|_2^2$. If one sets $\delta = \frac{4\nu}{m}$ then (3.2.8) can be improved as

$$\partial_t\|u'\|_2^2 \leq -2\nu\|u_{xx}\|_2^2 + 2\gamma\|u'\|_2^2 + m\int_0^\infty (\frac{|u|^6}{2\delta} + \frac{|u_{xx}|}{2}\delta)dx \tag{3.2.18}$$

$$\leq -2\nu\|u_{xx}\|_2^2 + 2\gamma\|u'\|_2^2 + \frac{m^2}{8\nu}\|u\|_6^6 + 2\nu\|u_{xx}\|_2^2$$

$$= 2\gamma\|u'\|_2^2 + \frac{m^2}{8\nu}\|u\|_6^6 \leq 2\gamma\|u'\|_2^2 + \frac{m^2}{8\nu}\lambda\|u'\|_2^2\|u\|_2^4$$

via (3.2.10). Now use $\|u\|_2^2 \leq \|u_0\|_2^2$ to obtain

$$\partial_t\|u'\|_2^2 \leq 2\gamma\|u'\|_2^2 + \frac{m^2}{8\nu}\lambda\|u'\|_2^2\|u_0\|_2^4 \leq 0 \tag{3.2.19}$$

Thus (3.2.19) implies $\|u'\|_2^2$ is bounded and by (3.2.14) one concludes $\|u\|_\infty$ is bounded for all $0 \leq x, t < \infty$. Q.E.D.

REMARK 3.2.4. *By Corollary (3.2.3) one notes in case* $\gamma < 0$ *there is a small initial norm condition* $\|u_0\|_2 \leq 2(\frac{\nu|\gamma|}{m^2\lambda})^{\frac{1}{4}}$ *that promises uniform boundedness of* $\|u\|_2, \|u'\|_2$ *and* $\|u\|_\infty$. *As we stated before, the initial-value problem for the Ginzburg-Landau equation with finite domain and zero boundary condition was solved in [27] but with approach other than semi-group technique here and the results are slightly different.*

Chapter 4.
THE HALF-LINE PROBLEM FOR THE GINZBURG-LANDAU EQUATION

In this chapter, we shall investigate the solvability of the initial-boundary value problem (so-called half-line problem) for the Ginzburg-Landau equation ($0 \le x < \infty, 0 \le t < \infty$) with initial and boundary data $u_0(0) = Q(0), u_0 \in H^2[0,\infty), Q(t) \in C^2[0,\infty)$. We shall in §4.1 prove a local existence-uniqueness theorem for the classical solution. In §4.2 we discuss the small amplitude solution to the half-line problem and give a criteria in terms of small initial-boundary data that one obtains a small amplitide solution (thus blow-up will not occur) on $[0,T]$. In §4.3 and §4.4 we prove that for $|\beta| \le \sqrt{3}\kappa$ or $\alpha\beta > 0$, this local solution is also a global one.

§4.1 LOCAL EXISTENCE THEOREM.

For the following Ginzburg-Landau equation

$$u_t = (\nu + i\alpha)u_{xx} - (\kappa + i\beta)|u|^2 u + \gamma u \qquad (4.1.1)$$

we assume that $\nu, \kappa > 0, \alpha \neq 0$ and all the other parameters β, κ are real. As in NLS case (cf. [18]), we shall first use the standard technique of change of variables via $u = v + Q(t)e^{-x}$. Thus this substitution in (4.1.1) yields

$$v_t = (\nu + i\alpha)v_{xx} - (\kappa + i\beta)|v|^2 v + G_0 + G_1 + G_2 \qquad (4.1.2)$$

$$v(0,t) = 0, v(x,0) = u_0(x) - Q(0)e^{-x} = v_0(x) \in H^2[0,\infty),$$

Here

$$(4.1.3) \qquad G_0 = (\gamma + \nu + i\alpha)Qe^{-x} - Q'e^{-x} - (\kappa + i\beta)|Q|^2Qe^{-3x}$$

$$(4.1.4) \qquad G_1 = -(\kappa + i\beta)(Q^2e^{-2x}\bar{v} + 2|Q|^2e^{-2x}v)$$

$$(4.1.5) \qquad G_2 = -(\kappa + i\beta)(2Qe^{-x}|v|^2 + \bar{Q}e^{-x}v^2)$$

For any $T > 0$, $t \in [0, T]$, clearly $G_0 = G_0(Q, Q', e^{-x})$ is uniformly bounded and Lipschitz in t with values in H^2; $\|G_1\|_{2,2} \leq c\|v\|_{2,2}$ (and evidently G_1 satisfies local Lipschitz condition). Again by $\|G_2\|_{2,2} \leq c\|v\|_\infty\|v\|_{2,2}$ one readily concludes that G_2 satisfies local Lipschitz condition also. Now we work with a semigroup generator $A = (\nu + i\alpha)D_x^2$ with $D(A) = \{f \in H^2[0,\infty); f(0) = 0\}$. One notices $D(A)$ is a dense subspace of $W = \{u \in L^2, u_{xx} \in L^2\}$ with a norm equivalent to H^2 norm. To check A is a semigroup generator, one looks at the resolvant $R_\lambda = (\lambda - A)^{-1}$ so consider $(\lambda - A)v = f$ with $v(0) = 0, f \in L^2$. Note that

$$(4.1.6) \qquad ((\lambda - A)v, v) = \int_0^\infty (\lambda v - (\nu + i\alpha)v_{xx})\bar{v}dx = \lambda\|v\|_2^2 + (\nu + i\alpha)\|v'\|_2^2$$

From (4.1.6) one has $|((\lambda - A)v, v)| \geq \min\{\lambda, \nu\}\|v\|_{H_0^1}^2$ and by Theorem 2.3.3 of [18], the operator $\lambda - A$ maps $D(A)$ 1-1 onto L^2 for $\lambda > 0$. The resolvent set of A contains R^+. Now take the real part and using Cauchy-Schwartz inequality one has

$$(4.1.7) \qquad Re((\lambda - A)v, v) = \lambda\|v\|_2^2 + \nu\|v'\|_2^2 \geq \lambda\|v\|_2^2$$

$$(4.1.8) \qquad \|(\lambda - A)^{-1}\| \leq \frac{1}{\lambda}$$

By Hille-Yosida Theorem the operator A generates a strongly continuous contraction semigroup $N(t) = \exp At$ (cf. [53]). One has generally $G_1, G_2 \in D(A)$ but $G_0 \notin D(A)$. However, since we assume that $Q \in C^2$, it is clear that $G_0'(t)$ is continuous from (4.1.3). By [53] one has $\int_0^t N(t-s)G_0(s)ds \in D(A)$. Thus we can then converts (4.1.1) to an intergral equation:

$$v(t) = N(t)v_0 + \int_0^t N(t-s)G_0(s)ds + \int_0^t N(t-s)(G_1(s) + G_2(s) - (\kappa + i\beta)|v|^2 v)ds \tag{4.1.9}$$

By same arguments in [18], one has

THEOREM 4.1.1. *Given $\nu, \kappa > 0, \alpha \neq 0, u_0 \in H^2[0,\infty), Q \in C^2[0,\infty), Q(0) = u_0(0)$, the equation (4.1.2) for v (hence (4.1.1) for u) has a unique classical solution v (and u) in $C^1([0,T_M), L^2[0,\infty)) \cap C^0([0,T_M), H^2[0,\infty))$ with either $T_M = \infty$ or $\lim_{t\to T_M} \|v\|_{2,2} = \infty$.*

§4.2 SMALL AMPLITUDE SOLUTION.

Unlike NLS, the global existence of solution to the half-line problem for the GL equation (4.1.1) generally is unknown. However one could study the small amplitude solution on any fixed interval $[0,T]$. These solutions are close to zero at each point in the whole space (with $t \leq T$). Such solutions arise in many physical peoblems when one perturbs around the zero state then one is intersted in the small-amplitude solutions of the perturbation equation (cf. [57]). We shall use the classical method to determine the criteria of small initial-boundary condition that promises the existence of small amplitude solution of the GL equation on any finite interval. Though the solution obtained is not global, it does eliminate a possible blow-up on $[0,T]$.

In this section we shall assume the initial data and boundary data are sufficiently small on fixed $[0,T]$ with certain norm. We try to prove that $\|u\|_2$ and $\|u'\|_2$ are small on $[0,T]$ thus to obtain a small amplitude solution and to prevent a blow-up. Let $\epsilon > 0$ be small and

$$\|Q\|_{H^1[0,T]} = \left(\int_0^T (|Q|^2 + |Q'|^2) d\tau\right)^{\frac{1}{2}} \leq \epsilon, \|u_0\|_{1,2} \leq \epsilon \tag{4.2.1}$$

We need to prove following estimates similar to (3.1) of [18].

LEMMA 4.2.1. *For $Q \in C^1$ one has*

(i) $\|u\|_2^2 = \|u_0\|_2^2 + \int_0^t (-2Re(\nu + i\alpha)P(\tau)\bar{Q}(\tau) - 2\nu\|u'\|_2^2 - 2\kappa\|u\|_4^4 + 2\gamma\|u\|_2^2) d\tau$

(ii)
$$\|u'\|_2^2 = \|u_0'\|_2^2 + \int_0^t (-2ReQ'(\tau)\bar{P}(\tau) - 2\gamma ReQ(\tau)\bar{P}(\tau)) d\tau$$

$$+ \int_0^t [-2\nu\|u_{xx}\|_2^2 + 2\gamma\|u'\|_2^2 + 2\kappa Re \int_0^\infty |u|^2 u\bar{u}_{xx} dx - 2\beta Im \int_0^\infty |u|^2 u\bar{u}_{xx} dx] d\tau$$

(iii)
$$\int_0^\infty u\bar{u}' dx = \int_0^\infty u_0\bar{u}_0' dx - \int_0^t Q(\tau)\overline{Q'(\tau)} d\tau$$

$$-i\alpha \int_0^t |P(\tau)|^2 d\tau + i\frac{1}{2}\beta \int_0^t |Q(\tau)|^4 d\tau$$

$$+ \int_0^t [\nu 2iIm \int_0^\infty u_{xx}\bar{u}_x dx - \kappa 2iIm \int_0^\infty |u|^2 u\bar{u}_x dx + \gamma 2iIm \int_0^\infty u\bar{u}_x] d\tau$$

PROOF: First by (4.1.1) and intergrating by parts

$$\partial_t \|u\|_2^2 = \int_0^\infty (u_t\bar{u} + \bar{u}_t u) dx \tag{4.2.2}$$

$$= \int_0^\infty [(\nu + i\alpha)u_{xx}\bar{u} - (\kappa + i\beta)|u|^4 + \gamma|u|^2]dx$$

$$+ \int_0^\infty [u(\nu - i\alpha)\bar{u}_{xx} - u(\kappa - i\beta)|u|^2\bar{u} + \gamma|u|^2]dx$$

$$= (\nu + i\alpha)u_x\bar{u} \Big|_0^\infty - (\nu + i\alpha)\|u'\|_2^2 + (\nu - i\alpha)\bar{u}_x u \Big|_0^\infty - (\nu - i\alpha)\|u'\|_2^2$$

$$-(\kappa - i\beta)\|u\|_4^4 - (\kappa + i\beta)\|u\|_4^4 + 2\gamma\|u\|_2^2$$

$$= -(\nu + i\alpha)P(t)\bar{Q}(t) - (\nu - i\alpha)\bar{P}(t)Q(t) - 2\nu\|u'\|_2^2 - 2\kappa\|u\|_4^4 + 2\gamma\|u\|_2^2$$

$$= -2Re(\nu + i\alpha)P(t)\bar{Q}(t) - 2\nu\|u'\|_2^2 - 2\kappa\|u\|_4^4 + 2\gamma\|u\|_2^2$$

Thus

$$(4.2.3) \quad \|u\|_2^2 = \|u_0\|_2^2 + \int_0^t (-2Re(\nu+i\alpha)P(\tau)\bar{Q}(\tau) - 2\nu\|u'\|_2^2 - 2\kappa\|u\|_4^4 + 2\gamma\|u\|_2^2)d\tau$$

This is (i). For (ii), one uses intergration by parts to find

$$(4.2.4) \qquad \partial_t\|u'\|_2^2 = \int_0^\infty (u_{xt}\bar{u}_x + u_x\bar{u}_{xt})dx$$

$$= u_t\bar{u}_x \Big|_0^\infty - \int_0^\infty u_t\bar{u}_{xx}dx + u_x\bar{u}_t \Big|_0^\infty - \int_0^\infty u_{xx}\bar{u}_t dx$$

$$= -Q'(t)\bar{P}(t) - \int_0^\infty [(\nu + i\alpha)|u_{xx}|^2 - (\kappa + i\beta)|u|^2 u\bar{u}_{xx} + \gamma u\bar{u}_{xx}]dx$$

$$-P(t)\bar{Q}'(t) - \int_0^\infty [(\nu - i\alpha)|u_{xx}|^2 - (\kappa - i\beta)|u|^2\bar{u}u_{xx} + \gamma u\bar{u}_{xx}]dx$$

$$= -2ReP(t)\bar{Q}'(t) - 2\nu\|u_{xx}\|_2^2 + 2\kappa Re\int_0^\infty |u|^2 u\bar{u}_{xx}dx - 2\beta Im\int_0^\infty |u|^2 u\bar{u}_{xx}dx$$

$$-\gamma u\bar{u}_x \Big|_0^\infty + \gamma\|u'\|_2^2 - \gamma\bar{u}u_x \Big|_0^\infty + \gamma\|u'\|_2^2$$

$$= -2ReP(t)\bar{Q}'(t) - 2\nu\|u_{xx}\|_2^2 + 2\kappa Re\int_0^\infty |u|^2 u\bar{u}_{xx}dx$$

$$-2\beta\int_0^\infty |u|^2 u\bar{u}_{xx}dx - 2\gamma ReP(t)\bar{Q}(t) + 2\gamma\|u'\|_2^2$$

Thus

$$\|u'\|_2^2 = \|u_0'\|_2^2 - 2\int_0^t ReQ'(\tau)\bar{P}(\tau)d \tag{4.2.5}$$

$$+\int_0^t (-2\gamma ReQ(\tau)\bar{P}(\tau))d\tau$$

$$+\int_0^t [-2\nu\|u_{xx}\|_2^2 + 2\gamma\|u'\|_2^2 + 2\kappa Re\int_0^\infty |u|^2 u\bar{u}_{xx}dx - 2\beta Im\int_0^\infty |u|^2 u\bar{u}_{xx}dx]d\tau$$

and we proved (ii). Finally for (iii),

$$\partial_t \int_0^\infty u\bar{u}_x dx = \int_0^\infty (u_t\bar{u}_x + u\bar{u}_{tx})dx \tag{4.2.6}$$

$$= \int_0^\infty u_t\bar{u}_x dx + u\bar{u}_t \Big|_0^\infty - \int_0^\infty u_x\bar{u}_t dx$$

$$= -Q(t)\bar{Q}'(t) + \int_0^\infty [(\nu + i\alpha)u_{xx}\bar{u}_x - (\kappa + i\beta)|u|^2 u\bar{u}_x + \gamma u\bar{u}_x]dx$$

$$-\int_0^\infty [(\nu - i\alpha)\bar{u}_{xx}u_x - (\kappa - i\beta)|u|^2\bar{u}u_x + \gamma\bar{u}u_x]dx$$

$$= -Q\bar{Q}' + 2\nu i Im\int_0^\infty u_{xx}\bar{u}_x dx + 2\alpha i Re\int_0^\infty u_{xx}\bar{u}_x dx$$

$$-2\kappa i Im\int_0^\infty |u|^2 u\bar{u}_x dx - 2\beta i Re\int_0^\infty |u|^2 u\bar{u}_x dx + 2\gamma i Im\int_0^\infty u\bar{u}_x dx$$

$$= -Q\bar{Q}' + 2\nu i Im\int_0^\infty u_{xx}\bar{u}_x dx + \alpha i\int_0^\infty \partial_x|u_x|^2 dx$$

$$-2\kappa i Im\int_0^\infty |u|^2 u\bar{u}_x dx - \beta i\int_0^\infty |u|^2\partial_x|u|^2 dx + 2\gamma i Im\int_0^\infty u\bar{u}_x dx$$

$$= -Q\bar{Q}' + 2\nu i Im\int_0^\infty u_{xx}\bar{u}_x dx - \alpha i|P|^2$$

$$-2\kappa i Im\int_0^\infty |u|^2 u\bar{u}_x dx + \frac{1}{2}\beta i|Q|^4 + 2\gamma i Im\int_0^\infty u\bar{u}_x dx$$

Therefore

$$\int_0^\infty u\bar{u}' dx = \int_0^\infty u_0\bar{u}_0' dx - \int_0^t Q(\tau)\overline{Q'(\tau)}d\tau \tag{4.2.7}$$

$$-i\alpha\int_0^t |P(\tau)|^2 d\tau + i\frac{1}{2}\beta\int_0^t |Q(\tau)|^4 d\tau$$

$$+\int_0^t [\nu 2iIm\int_0^\infty u_{xx}\bar{u}_x dx - \kappa 2iIm\int_0^\infty |u|^2 u\bar{u}_x dx + \gamma 2iIm\int_0^\infty u\bar{u}_x]d\tau$$

The proof is completed. Q.E.D.

Note Lemma 4.2.1 is very similar to Lemma 3.1 of [18], but with more complicated terms because of the presence of ν and κ terms in the GL equation. By (4.2.1) and the estimates in [50], there esists constants $\lambda > 0, a > 0, b > 0$ such that

$$\|u\|_6^6 \leq \lambda\|u'\|_2^2\|u\|_2^4 \tag{4.2.8}$$

$$\int_0^T |Q|^4 d\tau \leq a(\int_0^T |Q'|^2 d\tau)^{\frac{1}{2}}(\int_0^T |Q|^2 d\tau)^{\frac{3}{2}} + b(\int_0^T |Q|^2 d\tau)^{\frac{1}{2}} \leq a\epsilon^4 + b\epsilon \tag{4.2.9}$$

We shall assume that $\gamma = 0$ in this section for similicity (otherwise the estimates will become more complicated but does not affect the result). Let $0 \leq t \leq T$. By (4.2.1), (4.2.3) and the Cauchy-Schwartz inequality,

$$\begin{aligned}\|u\|_2^2 &\leq \|u_0\|_2^2 + 2(\nu + |\alpha|)(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}}(\int_0^t |Q(\tau)|^2 d\tau)^{\frac{1}{2}} \\ &\quad - \int_0^t (2\nu\|u'\|_2^2 + 2\kappa\|u\|_4^4)d\tau \\ &\leq \epsilon^2 + 2(\nu + |\alpha|)(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}}\epsilon - 2\nu\int_0^t \|u'\|_2^2 d\tau\end{aligned} \tag{4.2.10}$$

By (4.2.5),

$$\begin{aligned}\|u'\|_2^2 &\leq \|u_0'\|_2^2 + 2(\kappa + |\beta|)\int_0^t \|u\|_6^3\|u_{xx}\|_2 d\tau \\ &\quad -2\nu\int_0^t \|u_{xx}\|_2^2 d\tau + 2(\int_0^t |Q'(\tau)|^2 d\tau)^{\frac{1}{2}}(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}}\end{aligned} \tag{4.2.11}$$

$$\leq \epsilon^2 + 2\epsilon(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}} - 2\nu \int_0^t \|u_{xx}\|_2^2 d\tau + (\kappa + |\beta|) \int_0^t (\frac{\|u\|_6^6}{\delta} + \delta\|u_{xx}\|_2^2) d\tau$$

$$= \epsilon^2 + 2\epsilon(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}} - \nu \int_0^t \|u_{xx}\|_2^2 d\tau + \frac{\kappa + |\beta|}{\delta} \int_0^t \|u\|_6^6 d\tau$$

Here $\delta = \frac{\nu}{\kappa + |\beta|}$. By (4.2.8), (4.2.11) becomes

$$\|u'\|_2^2 \leq \epsilon^2 + 2\epsilon(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}} \tag{4.2.12}$$

$$-\nu \int_0^t \|u_{xx}\|_2^2 d\tau + \frac{\kappa + |\beta|}{\delta} \lambda \int_0^t \|u'\|_2^2 \|u\|_2^4 d\tau$$

Combine (4.2.10) and (4.2.12)

$$\|u\|_{1,2}^2 = \|u\|_2^2 + \|u'\|_2^2 \leq 2\epsilon^2 + 2(\nu + |\alpha| + 1)\epsilon(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}} \tag{4.2.13}$$

$$-2\nu \int_0^t \|u'\|_2^2 d\tau + \frac{\kappa + |\beta|}{\delta} \lambda \int_0^t \|u'\|_2^2 \|u\|_2^4 d\tau - \nu \int_0^t \|u_{xx}\|_2^2 d\tau$$

$$\leq 2\epsilon^2 + \epsilon\hat{c}(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}} + \int_0^t [-2\nu\|u'\|_2^2 + \frac{\tilde{c}}{2}\|u\|_{1,2}^6 - \nu\|u_{xx}\|_2^2] d\tau$$

Here

$$\hat{c} = 2(1 + \nu + |\alpha|), \tilde{c} = 2\frac{\kappa + |\beta|}{\delta}\lambda, \delta = \frac{\nu}{\kappa + |\beta|} \tag{4.2.14}$$

By (4.2.7),

$$|\alpha| \int_0^t |P(\tau)|^2 d\tau \leq |\int_0^\infty u_0 \bar{u}_0' dx| + |\int_0^\infty u\bar{u}' dx| + \frac{|\beta|}{2} \int_0^t |Q(\tau)|^4 d\tau \tag{4.2.15}$$

$$+ |\int_0^t Q(\tau)\overline{Q'(\tau)} d\tau| + 2\nu \int_0^t \|u_{xx}u_x\|_1 d\tau + 2\kappa \int_0^t \|u^3 u_x\|_1 d\tau$$

Apply (4.2.8),(4.2.9) to (4.2.15)

$$(4.2.16) \qquad |\alpha|\int_0^t |P(\tau)|^2 d\tau \le \frac{1}{2}|\beta|\int_0^t |Q(\tau)|^4 d\tau + \|u_0\|_2\|u_0'\|_2 + \|u\|_2\|u'\|_2$$

$$+2\nu\int_0^t \|u_{xx}\|_2\|u'\|_2 d\tau + 2\kappa\int_0^t \|u\|_6^3\|u'\|_2 d\tau + (\int_0^t |Q(\tau)|^2 d\tau)^{\frac{1}{2}}(\int_0^t |Q'(\tau)|^2 d\tau)^{\frac{1}{2}}$$

$$\le \frac{1}{2}|\beta|(a\epsilon^4 + b\epsilon) + \frac{1}{2}(\|u_0\|_2^2 + \|u_0'\|_2^2) + \frac{1}{2}(\|u\|_2^2 + \|u'\|_2^2)$$

$$+2\nu\int_0^t \|u_{xx}\|_2\|u'\|_2 d\tau + 2\kappa\int_0^t \sqrt{\lambda}\|u'\|_2\|u\|_2^2\|u'\|_2 d\tau + \epsilon^2$$

$$\le \frac{1}{2}|\beta|(a\epsilon^4 + b\epsilon) + \frac{1}{2}\epsilon^2 + \epsilon^2 + \frac{1}{2}\|u\|_{1,2}^2 + 2\nu\int_0^t \|u_{xx}\|_2\|u'\|_2 d\tau + 2\kappa\sqrt{\lambda}\int_0^t \|u\|_{1,2}^4 d\tau$$

Since $\epsilon \ll 1$, one can, without loss of generality, assume that $\epsilon \le 1$. Then (4.2.16) becomes

$$(4.2.17) \qquad (\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}} \le [\frac{|\beta|}{|\alpha|}(a\epsilon^4 + b\epsilon) + \frac{3\epsilon^2}{2|\alpha|} + \frac{\|u\|_{1,2}^2}{2|\alpha|}$$

$$+\frac{2\nu}{|\alpha|}\int_0^t \|u_{xx}\|_2\|u'\|_2 d\tau + \frac{2\kappa\sqrt{\lambda}}{|\alpha|}\int_0^t \|u\|_{1,2}^4 d\tau]^{\frac{1}{2}}$$

$$\le [\frac{|\beta|}{|\alpha|}(a+b)\epsilon + \frac{3\epsilon}{2|\alpha|} + \frac{\|u\|_{1,2}^2}{2|\alpha|} + c'\int_0^t \|u_{xx}\|_2\|u'\|_2 d\tau + c_0\int_0^t \|u\|_{1,2}^4 d\tau]^{\frac{1}{2}}$$

$$\le \bar{c}\sqrt{\epsilon} + \frac{\|u\|_{1,2}}{\sqrt{2|\alpha|}} + (c'\int_0^t \|u_{xx}\|_2\|u'\|_2 d\tau)^{\frac{1}{2}} + (c_0\int_0^t \|u\|_{1,2}^4 d\tau)^{\frac{1}{2}}$$

Here

$$(4.2.18) \qquad \bar{c} = \frac{|\beta|}{|\alpha|}(a+b) + \frac{3}{2|\alpha|}, c' = \frac{2\nu}{|\alpha|}, c_0 = \frac{2\kappa\sqrt{\lambda}}{|\alpha|}$$

Put (4.2.17) in (4.2.13) ($\epsilon \ll 1$)

$$(4.2.19) \qquad \|u\|_{1,2}^2 \le 2\epsilon^2 + \epsilon\hat{c}\{\bar{c}\sqrt{\epsilon} + \frac{\|u\|_{1,2}}{\sqrt{2|\alpha|}} + (c'\int_0^t \|u_{xx}\|_2\|u'\|_2 d\tau)^{\frac{1}{2}}\}$$

$$+\epsilon\hat{c}(c_0\int_0^t \|u\|_{1,2}^4 d\tau)^{\frac{1}{2}} + \int_0^t (-2\nu\|u'\|_2^2 + \frac{1}{2}\tilde{c}\|u\|_{1,2}^6 - \nu\|u_{xx}\|_2)d\tau$$

$$\leq 2\epsilon^2 + \hat{c}\bar{c}\epsilon^{\frac{3}{2}} + \frac{1}{2}(\frac{\epsilon\hat{c}}{\sqrt{2|\alpha|}})^2 + \frac{1}{2}\|u\|_{1,2}^2 + \frac{1}{2}\frac{(\epsilon\hat{c})^2 c'}{\nu} + \frac{1}{2}\nu\int_0^t \|u_{xx}\|_2\|u'\|_2 d\tau$$

$$+\frac{(\epsilon\hat{c})^2 c_0}{2} + \frac{1}{2}\int_0^t \|u\|_{1,2}^4 d\tau + \int_0^t (-2\nu\|u'\|_2^2 + \frac{1}{2}\tilde{c}\|u\|_{1,2}^6 - \nu\|u_{xx}\|_2^2)d\tau$$

$$\leq \frac{M}{2}\epsilon^{\frac{3}{2}} + \frac{1}{2}\|u\|_{1,2}^2 + \frac{\nu}{4}\int_0^t (\|u_{xx}\|_2^2 + \|u'\|_2^2)d\tau$$

$$+\frac{1}{2}\int_0^t \|u\|_{1,2}^4 + \int_0^t (-2\nu\|u'\|_2^2 + \frac{1}{2}\tilde{c}\|u\|_{1,2}^6 - \nu\|u_{xx}\|_2^2)d\tau$$

$$\leq \frac{M}{2}\epsilon^{\frac{3}{2}} + \frac{1}{2}\|u\|_{1,2}^2 + \frac{1}{2}\int_0^t \|u\|_{1,2}^4 d\tau + \frac{1}{2}\tilde{c}\int_0^t \|u\|_{1,2}^6 d\tau$$

Here

(4.2.20) $$M = 4 + 2\hat{c}\bar{c} + \frac{\hat{c}^2}{2|\alpha|} + \frac{\hat{c}^2 c'}{\nu} + \hat{c}^2 c_0$$

Rearrange (4.2.19)

(4.2.21) $$\|u\|_{1,2}^2 \leq M\epsilon^{\frac{3}{2}} + \int_0^t (\|u\|_{1,2}^4 + \tilde{c}\|u\|_{1,2}^6)d\tau$$

Let $M(t) = \sup_{0\leq t'\leq t}\|u\|_{1,2}^2$ then (4.2.21) implies (noting $0 \leq t \leq T$)

(4.2.22) $$M(t) \leq M\epsilon^{\frac{3}{2}} + \sup_{0\leq t'\leq t}\int_0^{t'} (\|u\|_{1,2}^4 + \tilde{c}\|u\|_{1,2}^6)d\tau$$

$$\leq M\epsilon^{\frac{3}{2}} + \int_0^t (M^2(\tau) + \tilde{c}M^3(\tau))d\tau \leq M\epsilon^{\frac{3}{2}} + TM^2(t) + \tilde{c}TM^3(t)$$

We shall claim that if $\epsilon < (\frac{1}{4TM+8TM^2\tilde{c}})^{\frac{2}{3}}, \epsilon \leq 1$ then our solution u on $[0,T]$ is a small amplitude solution with $\|u\|_{1,2}^2 < 2M\epsilon^{\frac{3}{2}}$. Note $M(t)$ is a continuous function with $M(0) \leq M\epsilon^{\frac{3}{2}}$ by (4.2.22). If our claim does not hold then by continuity of $M(t)$ one can find $t \in [0,T]$ such that $M(t) = 2M\epsilon^{\frac{3}{2}}$. Thus by (4.2.22)

$$(4.2.23) \qquad 2M\epsilon^{\frac{3}{2}} = M(t) \leq M\epsilon^{\frac{3}{2}} + TM^2(t) + \tilde{c}TM^3(t)$$

$$= M\epsilon^{\frac{3}{2}} + T(2M\epsilon^{\frac{3}{2}})^2 + \tilde{c}T(2M\epsilon^{\frac{3}{2}})^3$$

Thus (noting $\epsilon \leq 1$)

$$(4.2.24) \qquad 1 \leq 2T(2M\epsilon^{\frac{3}{2}}) + 2\tilde{c}T(2M\epsilon^{\frac{3}{2}})^2 \leq 4TM\epsilon^{\frac{3}{2}} + 8TM^2\tilde{c}(\epsilon^{\frac{3}{2}})^2$$

$$\leq 4TM\epsilon^{\frac{3}{2}} + 8TM^2\tilde{c}\epsilon^{\frac{3}{2}} = (4TM + 8TM^2\tilde{c})\epsilon^{\frac{3}{2}} < 1$$

which is a contradiction. Now by estimates in [50], there exists $\tilde{\lambda} > 0$ such that

$$(4.2.25) \qquad \|u\|_\infty \leq \tilde{\lambda}\|u'\|_2^{\frac{1}{2}}\|u\|_2^{\frac{1}{2}} \leq \frac{1}{2}\tilde{\lambda}(\|u'\|_2 + \|u\|_2)$$

$$\leq \tilde{\lambda}\sqrt{M(t)} \leq \tilde{\lambda}\sqrt{2M\epsilon^{\frac{3}{2}}} < \infty$$

By similar estimates like (4.2), (4.3) of [18] and Gronwall lemma (see the proof of Theorem 4.1 in [18]), one has $\|u\|_{2,2} \leq c$ on $[0,T]$. Thus we have proved the following result

THEOREM 4.2.2 (SMALL AMPLITUDE SOLUTION). *If the initial-boundary data* $\|u_0\|_{1,2} \leq \epsilon, \|Q\|_{H^1[0,T]} \leq \epsilon$ *with* $\gamma = 0, \epsilon \leq 1, \epsilon < (\frac{1}{4TM+8TM^2\tilde{c}})^{\frac{2}{3}}$ *where* $M, \tilde{c}$ *are given by (4.2.20) and (4.2.14), then the unique classical solution of the GL equation (4.1.1)* $u \in C^0(H^2[0,\infty)) \cap C^1(L^2[0,\infty))$ *exists for* $t \in [0,T]$. *Further, this solution has small amplitude* $\|u\|_{1,2} < 2M\epsilon^{\frac{3}{2}}$ *and* $\|u\|_\infty \leq \tilde{\lambda}\sqrt{2M\epsilon^{\frac{3}{2}}}$ *on [0,T] where* $\tilde{\lambda}$ *is determined by (4.2.25).*

We shall note that small initial-boundary data on $[0,T]$ will produce a small amplitude solution on $[0,T]$ to eliminate a blow-up. But it remains a question if this solution is a global one. However, we shall show in the next section that for certain GL equation, global solution does exist.

§4.3 GLOBAL EXISTENCE THEOREM (I).

We shall start to show the global existence of the half-line problem for the GL equation in case $|\beta| \leq \sqrt{3}\kappa$. To accomplish this we shall prove that for any interval $[0,T]$, the norm $\|u\|_\infty$ is bounded to conclude that $\|u\|_{2,2}$ is bounded. Let $Q_0 = \|Q\|_{C^1[0,T]} = \sup_{0\leq t\leq T}(|Q(t)| + |Q'(t)|) < \infty$.

LEMMA 4.3.1. *For $|\beta| \leq \sqrt{3}\kappa, \nu, \kappa > 0, \alpha \neq 0, 0 \leq t \leq T$, there exists $\hat{c}, \tilde{c} > 0$ such that*

(i) $$\|u'\|_2^2 \leq [\|u_0'\|_2^2 + \hat{c}(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}} - 2\nu \int_0^t \|u_{xx}|_2^2 d\tau]e^{2|\gamma|T}$$

(ii) $$\|u\|_2^2 \leq [\|u_0\|_2^2 + \tilde{c}(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}}]e^{2|\gamma|T}$$

PROOF: First intergrate the following by parts:

(4.3.1) $$2\kappa Re \int_0^\infty |u|^2 u\bar{u}_{xx} dx - 2\beta Im \int_0^\infty |u|^2 u\bar{u}_{xx} dx$$

$$= 2\kappa Re|u|^2 u\bar{u}_x \Big|_0^\infty - 2\kappa Re \int_0^\infty (2uu_x\bar{u} + u^2\bar{u}_x)\bar{u}_x dx$$

$$-2\beta Im|u|^2 u\bar{u}_x \Big|_0^\infty + 2\beta Im \int_0^\infty (2uu_x\bar{u} + u^2\bar{u}_x)\bar{u}_x dx$$

$$= -2\kappa Re|Q(t)|^2 Q(t)\bar{P}(t) - 2\kappa Re \int_0^\infty (2|u|^2|u_x|^2 + u^2\bar{u}_x^2) dx$$

$$+2\beta Im|Q(t)|^2 Q(t)\bar{P}(t) + 2\beta Im \int_0^\infty (2|u|^2|u_x|^2 + u^2\bar{u}_x^2) dx$$

$$= -2\kappa Re|Q(t)|^2 Q(t)\bar{P}(t) + 2\beta Im|Q(t)|^2 Q(t)\bar{P}(t)$$

$$-2\kappa Re \int_0^\infty (2|u|^2|u_x|^2 + u^2 u_x^2) dx + 2\beta Im \int_0^\infty u^2\bar{u}_x^2 dx$$

By setting up $u\bar{u}_x = a + bi, u^2\bar{u}_x^2 = a^2 - b^2 + 2abi$ one observes

$$(4.3.2) \qquad B(t) = -2\kappa Re \int_0^\infty (2|u|^2|u_x|^2 + u^2\bar{u}_x^2)dx + 2\beta Im \int_0^\infty u^2\bar{u}_x^2 dx$$

$$= -2\kappa Re \int_0^\infty (2(a^2+b^2) + (a^2 - b^2 + 2abi))dx + 2\beta Im \int_0^\infty (a^2 - b^2 + 2abi)dx$$

$$= \int_0^\infty (-2\kappa(3a^2 + b^2) + 2\beta 2ab)dx = -2\kappa \int_0^\infty (3a^2 - \frac{2\beta ab}{\kappa} + b^2)dx$$

Since $|\beta| \leq \sqrt{3}\kappa, \Delta = (\frac{-2\beta b}{\kappa})^2 - 4 \times 3b^2 = \frac{4b^2}{\kappa^2}(\beta^2 - 3\kappa^2) \leq 0$, one concludes $3a^2 - \frac{2\beta ab}{\kappa} + b^2 \geq 0$ hence $B(t) \leq 0$ because $\kappa > 0$. Then go back to (4.2.5) and apply (4.3.1), (4.3.2) (the upshot here is $B(\tau) \leq 0$ for $\tau \geq 0$):

$$(4.3.3) \qquad \|u'\|_2^2 = \|u_0'\|_2^2 + \int_0^t (-2ReP\bar{Q}' - 2\gamma ReP\bar{Q} - 2\nu\|u_{xx}\|_2^2 + 2\gamma\|u'\|_2^2)d\tau$$

$$+ \int_0^t (2\kappa Re \int_0^\infty |u|^2 u\bar{u}_{xx}dx - 2\beta Im \int_0^\infty |u|^2 u\bar{u}_{xx}dx)d\tau$$

$$= \|u_0'\|_2^2 + \int_0^t (-2ReP\bar{Q}' - 2\gamma ReP\bar{Q} - 2\nu\|u_{xx}\|_2^2 + 2\gamma\|u'\|_2^2)d\tau$$

$$+ \int_0^t (-2\kappa Re|Q|^2 Q\bar{P} + 2\beta Im|Q|^2 Q\bar{P} + B(\tau))d\tau$$

$$\leq \|u_0'\|_2^2 + 2\int_0^t |P\bar{Q}'|d\tau + 2\int_0^t |\gamma P\bar{Q}|d\tau - 2\nu \int_0^t \|u_{xx}\|_2^2 d\tau$$

$$+2\gamma \int_0^t \|u'\|_2^2 d\tau + 2(\kappa + |\beta|)\int_0^t |Q|^3|\bar{P}|d\tau$$

$$\leq \|u_0'\|_2^2 + 2(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}}(\int_0^t |Q'(\tau)|^2 d\tau)^{\frac{1}{2}}$$

$$+2|\gamma|(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}}(\int_0^t |Q(\tau)|^2 d\tau)^{\frac{1}{2}}$$

$$-2\nu \int_0^t \|u_{xx}\|_2^2 d\tau + 2\gamma \int_0^t \|u'\|_2^2 d\tau + 2(\kappa + |\beta|)(\int_0^t |Q(\tau)|^6 d\tau)^{\frac{1}{2}}(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}}$$

$$\leq \|u_0'\|_2 + 2Q_0\sqrt{T}(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}} + 2|\gamma|Q_0\sqrt{T}(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}}$$

$$-2\nu \int_0^t \|u_{xx}\|_2^2 d\tau + 2\gamma \int_0^t \|u'\|_2^2 d\tau + 2(\kappa + |\beta|)Q_0^3\sqrt{T}(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}}$$

$$= \|u_0'\|_2^2 + \hat{c}(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}} - 2\nu \int_0^t \|u_{xx}\|_2^2 d\tau + 2\gamma \int_0^t \|u'\|_2^2 d\tau$$

We shall treat $c, c_0, \bar{c}, \tilde{c}, c', \hat{c}$ as generic constants. By Gronwall lemma $(0 \leq t \leq T)$

$$(4.3.4) \qquad \|u'\|_2^2 \leq [\|u_0'\|_2^2 + \hat{c}(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}} - 2\nu \int_0^t \|u_{xx}|_2^2 d\tau]e^{2|\gamma|T}$$

This is (i) of Lemma 4.3.1. For (ii), one goes to (4.2.3)

$$(4.3.5) \qquad \|u\|_2^2 \leq \|u_0\|_2^2 + 2(\nu + |\alpha|)(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}}(\int_0^t |Q(\tau)|^2 d\tau)^{\frac{1}{2}}$$

$$-2\kappa \int_0^t \|u\|_4^4 d\tau - 2\nu \int_0^t \|u'\|_2^2 d\tau + 2\gamma \int_0^t \|u\|_2^2 d\tau$$

$$\leq \|u_0\|_2^2 + 2(\nu + |\alpha|)Q_0\sqrt{T}(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}} + 2\gamma \int_0^t \|u\|_2^2 d\tau$$

$$= \|u_0\|_2^2 + \tilde{c}(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}} + 2\gamma \int_0^t \|u\|_2^2 d\tau$$

By Gronwall lemma,

$$(4.3.6) \qquad \|u\|_2^2 \leq [\|u_0\|_2^2 + \tilde{c}(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}}]e^{2|\gamma|T}$$

Thus Lemma 4.3.1 is proved. Q.E.D.

LEMMA 4.3.2. *$\|u\|_{1,2}$ and $\|u\|_\infty$ are bounded on $[0,T]$.*

PROOF: Add (4.3.4), (4.3.6) together

$$(4.3.7) \qquad \|u\|_{1,2}^2 \leq [\|u_0\|_{1,2}^2 + (\tilde{c} + \hat{c})(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}} - 2\nu \int_0^t \|u_{xx}\|_2^2 d\tau]e^{2|\gamma|T}$$

$$= m[\|u_0\|_{1,2}^2 + \bar{c}(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}} - 2\nu \int_0^t \|u_{xx}\|_2^2 d\tau]$$

Rewrite (4.2.7) as follows (using (4.2.8))

$$\text{(4.3.8)} \qquad |\alpha| \int_0^t |P(\tau)|^2 d\tau \le \|u_0\|_2 \|u_0'\|_2 + \|u\|_2 \|u'\|_2 + \frac{1}{2}|\beta| \int_0^t |Q(\tau)|^4 d\tau$$

$$+ |\int_0^t Q(\tau)\overline{Q'(\tau)} d\tau| + \int_0^t (2\nu \|u_{xx}\|_2 \|u'\|_2 + 2\kappa \|u^3\|_2 \|u'\|_2 + 2|\gamma| \|u\|_2 \|u'\|_2) d\tau$$

$$\le \frac{1}{2}\|u_0\|_{1,2}^2 + \frac{1}{2}\|u\|_{1,2}^2 + \frac{1}{2}|\beta| Q_0^4 T + Q_0^2 T + \int_0^t \nu(\|u_{xx}\|_2^2 + \|u'\|_2^2) d\tau$$

$$+ 2\kappa \int_0^t \sqrt{\lambda} \|u'\|_2 \|u\|_2^2 \|u'\|_2 d\tau + |\gamma| \int_0^t (\|u\|_2^2 + \|u'\|_2^2) d\tau$$

Thus

$$\text{(4.3.9)} \quad (\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}} \le [c_0 + \tilde{c}\|u\|_{1,2}^2 + \hat{c} \int_0^t (\|u_{xx}\|_2^2 + \|u\|_{1,2}^4 + \|u\|_{1,2}^2) d\tau]^{\frac{1}{2}}$$

$$\le [c_0 + \tilde{c}\|u\|_{1,2}^2 + \hat{c} \int_0^t (\|u_{xx}\|_2^2 + \|u\|_{1,2}^4 + \frac{1}{2}\|u\|_{1,2}^4 + \frac{1}{2}) d\tau]^{\frac{1}{2}}$$

$$\le [c_0 + \tilde{c}\|u\|_{1,2}^2 + \hat{c} \int_0^t \|u_{xx}\|_2^2 d\tau + c \int_0^t \|u\|_{1,2}^4 d\tau + \hat{c}\frac{1}{2}T]^{\frac{1}{2}}$$

$$\le \bar{c}_0 + c'\|u\|_{1,2} + (\hat{c} \int_0^t \|u_{xx}\|_2^2 d\tau)^{\frac{1}{2}} + (c \int_0^t \|u\|_{1,2}^4 d\tau)^{\frac{1}{2}}$$

Put (4.3.9) in (4.3.7) to get

$$\text{(4.3.10)} \qquad \|u\|_{1,2}^2 \le m(\|u_0\|_{1,2}^2 - 2\nu \int_0^t \|u_{xx}\|_2^2 d\tau)$$

$$+ m\bar{c}(\bar{c}_0 + c'\|u\|_{1,2} + (\hat{c} \int_0^t \|u_{xx}\|_2^2 d\tau)^{\frac{1}{2}} + (c \int_0^t \|u\|_{1,2}^4 d\tau)^{\frac{1}{2}})$$

$$\le m\|u_0\|_{1,2}^2 + m\bar{c}\bar{c}_0 - 2m\nu \int_0^t \|u_{xx}\|_2^2 d\tau$$

$$+\frac{1}{2}(m\bar{c}c')^2 + \frac{1}{2}\|u\|_{1,2}^2 + \frac{1}{2}\frac{(m\bar{c})^2\hat{c}}{4m\nu} + \frac{1}{2}(4m\nu)\int_0^t \|u_{xx}\|_2^2 d\tau + m(c\int_0^t \|u\|_{1,2}^4 d\tau)^{\frac{1}{2}}$$

$$= m' + \frac{1}{2}\|u\|_{1,2}^2 + m\bar{c}(c\int_0^t \|u\|_{1,2}^4 d\tau)^{\frac{1}{2}}$$

Thus

$$\|u\|_{1,2}^2 \le 2m' + 2m\bar{c}(c\int_0^t \|u\|_{1,2}^4 d\tau)^{\frac{1}{2}} \tag{4.3.11}$$

By squaring both sides of (4.3.11) one has

$$\|u\|_{1,2}^4 - 4m'\|u\|_{1,2}^2 + (2m')^2 = (\|u\|_{1,2}^2 - 2m')^2 \le 4m^2\bar{c}^2 c\int_0^t \|u\|_{1,2}^4 d\tau \tag{4.3.12}$$

Therefore

$$\|u\|_{1,2}^4 \le 4m'\|u\|_{1,2}^2 + 4m^2\bar{c}^2 c\int_0^t \|u\|_{1,2}^4 d\tau \tag{4.3.13}$$

$$\le \frac{1}{2}(4m')^2 + \frac{1}{2}\|u\|_{1,2}^4 + 4m^2\bar{c}^2 c\int_0^t \|u\|_{1,2}^4 d\tau$$

$$\|u\|_{1,2}^4 \le (4m')^2 + 8m^2\bar{c}^2 c\int_0^t \|u\|_{1,2}^4 d\tau$$

And by Gronwall lemma,

$$\|u\|_{1,2}^4 \le (4m')^2 e^{8m^2\bar{c}^2 ct} \le (4m')^2 e^{8m^2\bar{c}^2 cT} \tag{4.3.14}$$

Also

$$\|u\|_\infty \le \tilde{\lambda}\|u\|_2^{\frac{1}{2}}\|u'\|_2^{\frac{1}{2}} \le \tilde{\lambda}\|u\|_{1,2} \le \tilde{\lambda}\sqrt{4m'}e^{m^2\bar{c}^2 cT} \tag{4.3.15}$$

Thus $\|u\|_\infty$ is bounded on $[0,T]$. Q.E.D.

THEOREM 4.3.3 (GLOBAL EXISTENCE THEOREM (I)). *For $u_0(x) \in H^2, Q(t) \in C^2, u_0(0) = Q(0), |\beta| \le \sqrt{3}\kappa$, there exists a unique global classical solution $u \in C^0(H^2[0,\infty)) \cap C^1(L^2[0,\infty))$ to (4.1.1).*

PROOF: Let us go back to (4.1.9) for a moment. Set $w_0 = N(t)v_0 + \int_0^t N(t-s)G_0(s)ds \in D(A)$. Then

$$v(t) = w_0 + \int_0^t N(t-s)(G_1(s) + G_2(s) - (\kappa + i\beta)|v|^2 v)ds \tag{4.3.16}$$

Note N is a contraction semigroup in L^2 hence also in $D(A)$ in graph norm. Lemma 4.3.2 certainly implies that $\|v\|_\infty < \infty$. Thus by estimates on G_1, G_2 and (4.3.16) one has

$$\begin{aligned} \|v\|_{D(A)} &\leq \|w_0 + \int_0^t N(t-s)(G_1(s) + G_2(s) - (\kappa + i\beta)|v|^2 v)ds\|_{D(A)} \\ &\leq \|w_0\|_{D(A)} + \int_0^t \|N(t-s)(G_1(s) + G_2(s) - (\kappa + i\beta)|v|^2 v)\|_{D(A)} ds \\ &\leq c_1 + \int_0^t [c_2\|v\|_{2,2} + c_3\|v\|_\infty \|v\|_{2,2} + c_4\|v\|_\infty \|v\|_{2,2}]ds \\ &\leq c_1 + c_5 \int_0^t \|v\|_{2,2} ds \end{aligned} \tag{4.3.17}$$

Here we realize that $D(A)$ norm is equivalent to H^2 norm. By Gronwall lemma one concludes from (4.3.17) that $\|v\|_{2,2}$ and $\|u\|_{2,2}$ are bounded on $[0, T]$ for any $T > 0$. Now we have completed the proof of global existence. Q.E.D.

REMARK 4.3.4. *The following rescaled Ginzburg-Landau equation*

$$A_t = RA + (1 + i\nu)A_{xx} - (1 + i\mu)|A|^2 A \tag{4.3.18}$$

has been studied extensively (for example, see [6,21,22]) where $\nu = \epsilon a, \mu = \epsilon b$. It leads to a perturbation analysis on a complex Duffing equation. Since one assumes $\epsilon \ll 1$ the condition $|\beta| \leq \sqrt{3}\kappa$ is satisfied where $\beta = \epsilon b, \kappa = 1$. Hence Theorem 4.3.3 gives the global existence of the half-line problem (4.3.18). It should be noted that the criterion $|\mu| \leq \sqrt{3}$ had been established by several authors (cf.[6,22]), *but for different purposes. (It was shown that $\sqrt{3}$ is the critical value of μ for which there exists a ν such that the homogeneous rotating wave is stable to a sideband before the trivial solution goes unstable to the second rotating wave as R is increased from 0.)*

We expect that well-posedness of (4.1.1) holds in case $|\beta| \leq \sqrt{3}\kappa$ via similar method used in Chapter 1. to prove the well-posedness of the half-line problem for NLS.

§4.4 GLOBAL EXISTENCE THEOREM (II).

We shall prove the global existence for (4.1.1) if $\alpha\beta > 0$, i.e. α and β have the same sign. The estimates are a little different from those used in §4.3 but basic idea remains the same, i.e. to show that $\|u\|_{1,2}$ is bounded on any finite interval $[0, T]$. One application of our result will lead to global existence of following initial-boundary value problem: $u_t = (\epsilon \pm i)u_{xx} - (\epsilon \pm i)|u|^2 u$ which can be regarded as a perturbed NLS. Further, if u_ϵ solves the half-line problem for the GL equation then $u_\epsilon \to u$ under certain norm when $\epsilon \to 0$ where u solves the half-line NLS with the same initial-boundary data.

First some estimates. From (4.1.1) one has

(4.4.1)
$$u_t\bar{u}_t = (\nu + i\alpha)u_{xx}\bar{u}_t - (\kappa + i\beta)|u|^2 u\bar{u}_t + \gamma u$$

$$\bar{u}_t u_t = (\nu + i\alpha)\bar{u}_{xx}u_t - (\kappa - i\beta)|u|^2\bar{u}u_t + \gamma\bar{u}u_t$$

The difference is (after cancelling all i's)

(4.4.2)
$$0 = 2Im(\nu u_{xx} - \kappa|u|^2 u + \gamma u)\bar{u}_t + 2\alpha Re u_{xx}\bar{u}_t - 2\beta Re|u|^2 u\bar{u}_t$$

Integrate from 0 to ∞

$$(4.4.3) \qquad 0 = \int_0^\infty 2Im(\nu u_{xx} - \kappa|u|^2u + \gamma u)\bar{u}_t dx$$

$$+ \int_0^\infty 2\alpha Re u_{xx}\bar{u}_t dx - \int_0^\infty 2\beta Re|u|^2 u\bar{u}_t dx$$

$$= \int_0^\infty 2Im(\nu u_{xx} - \kappa|u|^2u + \gamma u)((\nu - i\alpha)\bar{u}_{xx} - (\kappa - i\beta)|u|^2\bar{u} + \gamma\bar{u})dx$$

$$+2\alpha Re u_x\bar{u}_t \Big|_0^\infty - 2\alpha Re \int_0^\infty u_x\bar{u}_{tx} dx - \beta \int_0^\infty |u|^2\partial_t|u|^2 dx$$

$$= \int_0^\infty 2Im[(\nu^2 - i\alpha\nu)|u_{xx}|^2 - \kappa(\nu - i\alpha)|u|^2u\bar{u}_{xx} + \gamma(\nu - i\alpha)u\bar{u}_{xx}]dx$$

$$+ \int_0^\infty 2Im[-\nu(\kappa - i\beta)|u|^2\bar{u}u_{xx} + \kappa(\kappa - i\beta)|u|^6 - \gamma(\kappa - i\beta)|u|^4 + \gamma\nu u_{xx}\bar{u}]dx$$

$$+ \int_0^\infty 2Im[-\kappa\nu|u|^4 + \gamma^2|u|^2]dx - 2\alpha Re P(t)\bar{Q}'(t) - \alpha\partial_t\|u'\|_2^2 - \frac{1}{2}\beta\partial_t\|u\|_4^4$$

Rearranging (4.4.3)

$$(4.4.4) \qquad \partial_t(\alpha\|u'\|_2^2 + \frac{1}{2}\beta\|u\|_4^4) = -2\alpha Re P\bar{Q}' + \int_0^\infty -2\alpha\nu|u_{xx}|^2 dx - 2\kappa\beta\|u\|_6^6$$

$$+2Im \int_0^\infty [\kappa(i\alpha - \nu)|u|^2u\bar{u}_{xx} + \nu(i\beta - \kappa)|u|^2\bar{u}u_{xx}]dx$$

$$+2Im \int_0^\infty [\gamma(\nu - i\alpha)u\bar{u}_{xx} + \gamma\nu u_{xx}\bar{u}]dx$$

Write

$$(4.4.5) \qquad |u|^2u\bar{u}_{xx} = A + Bi, |u|^2\bar{u}u_{xx} = A - Bi$$

then (4.4.4) becomes

$$(4.4.6) \qquad \partial_t(\alpha\|u'\|_2^2 + \frac{1}{2}\beta\|u\|_4^4) = -2\alpha Re P\bar{Q}' - 2\alpha\nu\|u_{xx}\|_2^2 - 2\kappa\beta\|u\|_6^6$$

$$+2Im \int_0^\infty [\kappa(i\alpha - \nu)(A + Bi) + \nu(i\beta - \kappa)(A - Bi)]dx$$

$$+2Im[\gamma(\nu - i\alpha)u\bar{u}_x + \gamma\nu u_x\bar{u}]\Big|_0^\infty - 2Im\int_0^\infty [\gamma(\nu - i\alpha)|u_x|^2 + \gamma\nu|u_x|^2]dx$$

$$= -2\alpha Re P\bar{Q}' - 2\alpha\nu\|u_{xx}\|_2^2 - 2\kappa\beta\|u\|_6^6$$

$$+2Im\int_0^\infty (A\kappa i\alpha - \kappa\nu A - B\alpha\kappa - \kappa\nu Bi + \nu i\beta A - \nu\kappa A + \nu\beta B + \nu\kappa Bi)dx$$

$$-2Im[\gamma(\nu - i\alpha)Q\bar{P} + \gamma\nu P\bar{Q}] + 2\gamma\alpha\|u'\|_2^2$$

$$= -2\alpha Re P\bar{Q}' - 2\alpha\nu\|u_{xx}\|_2^2 - 2\kappa\beta\|u\|_6^6 + 2\int_0^\infty (A\alpha\kappa + A\nu\beta)dx$$

$$-2Im\gamma\nu(Q\bar{P} + P\bar{Q}) - 2Im(-i\alpha\gamma Q\bar{P}) + 2\gamma\alpha\|u'\|_2^2$$

$$= -2\alpha Re P\bar{Q}' - 2\alpha\nu\|u_{xx}\|_2^2 - 2\kappa\beta\|u\|_6^6$$

$$+2(\kappa\alpha + \nu\beta)\int_0^\infty Re|u|^2u\bar{u}_{xx}dx + 2\alpha\gamma Re Q\bar{P} + 2\gamma\alpha\|u'\|_2^2$$

Hence

$$\partial_t(\alpha\|u'\|_2^2 + \frac{1}{2}\beta\|u\|_4^4) + 2\alpha\nu\|u_{xx}\|_2^2 + 2\kappa\beta\|u\|_6^6 \tag{4.4.7}$$

$$= -2\alpha Re P\bar{Q}' + 2\alpha\gamma Re Q\bar{P} + 2\gamma\alpha\|u'\|_2^2 + 2(\kappa\alpha + \nu\beta)\int_0^\infty Re|u|^2u\bar{u}_{xx}dx$$

$$= -2\alpha Re P\bar{Q}' + 2\alpha\gamma Re Q\bar{P} + 2\gamma\alpha\|u'\|_2^2$$

$$+2(\kappa\alpha + \nu\beta)[Re|u|^2u\bar{u}_x\Big|_0^\infty - Re\int_0^\infty (2uu_x\bar{u} + u^2\bar{u}_x)\bar{u}_x dx]$$

$$= -2\alpha Re P\bar{Q}' + 2\alpha\gamma Re Q\bar{P} + 2\gamma\alpha\|u'\|_2^2$$

$$+2(\kappa\alpha + \nu\beta)(-Re|Q|^2Q\bar{P}) - 2(\kappa\alpha + \nu\beta)\int_0^\infty (2|u|^2|u_x|^2 + Re u^2\bar{u}_x^2)dx$$

Now consider $0 \leq t \leq T$ for ant $T > 0$. Let $\alpha\beta > 0$. Without loss of generality assume in (4.4.7) that $\alpha > 0, \beta > 0$. Note $\nu > 0, \kappa > 0$ also. This implies that

$$-2(\kappa\alpha + \nu\beta)\int_0^\infty (2|u|^2|u_x|^2 + Re u^2\bar{u}_x^2)dx \leq 0 \tag{*}$$

Thus (4.4.7) becomes (via (*))

$$\partial_t(\alpha\|u'\|_2^2 + \frac{1}{2}\beta\|u\|_4^4) \leq -2\alpha\nu\|u_{xx}\|_2^2 + 2\alpha|P\bar{Q}'| \tag{4.4.8}$$

$$+2\alpha|\gamma||P\bar{Q}|+2\alpha|\gamma|\|u'\|_2^2+2(\kappa\alpha+\nu\beta)|Q|^3|\bar{P}|$$

(4.4.9) $$\alpha\|u'\|_2^2+\frac{1}{2}\beta\|u\|_4^4\le\alpha\|u_0'\|_2^2+\frac{1}{2}\beta\|u_0\|_4^4-2\alpha\nu\int_0^t\|u_{xx}\|_2^2d\tau$$

$$+\int_0^t[2\alpha|P\bar{Q}'|+2\alpha|\gamma||P\bar{Q}|+2(\kappa\alpha+\nu\beta)|Q|^3|\bar{P}|]d\tau+2\alpha|\gamma|\int_0^t\|u'\|_2^2d\tau$$

(4.4.10) $$\alpha\|u'\|_2^2\le\alpha\|u_0'\|_2^2+\frac{1}{2}\beta\|u_0\|_4^4-2\alpha\nu\int_0^t\|u_{xx}\|_2^2d\tau$$

$$+\int_0^t|P|[2\alpha Q_0+2\alpha|\gamma|Q_0+2(\kappa\alpha+\nu\beta)Q_0^3|]d\tau+2\alpha|\gamma|\int_0^t\|u'\|_2^2d\tau$$

Here $Q_0=\max_{0\le t\le T}(|Q|+|Q'|)$. Thus from (4.4.10)

(4.4.11) $$\|u'\|_2^2\le c_0-2\nu\alpha\int_0^t\|u_{xx}\|_2^2d\tau+\tilde{c}\int_0^t|P|d\tau+2|\gamma|\int_0^t\|u'\|_2^2d\tau$$

$$\le c_0-2\nu\alpha\int_0^t\|u_{xx}\|_2^2d\tau+\tilde{c}\sqrt{T}(\int_0^t|P(\tau)|^2d\tau)^{\frac{1}{2}}+2|\gamma|\int_0^t\|u'\|_2^2d\tau$$

By Gronwall lemma,

(4.4.12) $$\|u'\|_2^2\le[c_0-2\nu\alpha\int_0^t\|u_{xx}\|_2^2d\tau+\hat{c}(\int_0^t|P(\tau)|^2d\tau)^{\frac{1}{2}}]e^{2|\gamma|T}$$

Add (4.4.12) and (4.3.6) together,

(4.4.13) $$\|u\|_{1,2}^2\le[c_0+\|u_0\|_2^2-2\nu\alpha\int_0^t\|u_{xx}\|_2^2d\tau+(\tilde{c}+\hat{c})(\int_0^t|P(\tau)|^2d\tau)^{\frac{1}{2}}]e^{2|\gamma|T}$$

$$\le[\tilde{c}_0-2\nu\alpha\int_0^t\|u_{xx}\|_2^2d\tau+\bar{c}(\int_0^t|P(\tau)|^2d\tau)^{\frac{1}{2}}]m$$

Put (4.3.9) in (4.4.13) to obtain

(4.4.14) $$\|u\|_{1,2}^2\le m(\tilde{c}_0-2\nu\alpha\int_0^t\|u_{xx}\|_2^2d\tau)$$

$$+m\bar{c}(\bar{c}_0 + c'\|u\|_{1,2} + (\hat{c}\int_0^t \|u_{xx}\|_2^2 d\tau)^{\frac{1}{2}} + (c\int_0^t \|u\|_{1,2}^4 d\tau)^{\frac{1}{2}})$$

$$= m_0\tilde{c}_0 + m\bar{c}\bar{c}_0 - 2m\nu\alpha\int_0^t \|u_{xx}\|_2^2 d\tau$$

$$+\frac{1}{2}(m\bar{c}c')^2 + \frac{1}{2}\|u\|_{1,2}^2 + \frac{1}{2}\frac{(m\bar{c})^2\hat{c}}{4m\nu\alpha} + \frac{1}{2}(4m\nu\alpha)\int_0^t \|u_{xx}\|_2^2 d\tau + m\bar{c}(c\int_0^t \|u\|_{1,2}^4 d\tau)^{\frac{1}{2}}$$

$$= m' + \frac{1}{2}\|u\|_{1,2}^2 + m\bar{c}(c\int_0^t \|u\|_{1,2}^4 d\tau)^{\frac{1}{2}}$$

Thus

$$\|u\|_{1,2}^2 \le 2m' + 2m\bar{c}(c\int_0^t \|u\|_{1,2}^4 d\tau)^{\frac{1}{2}} \tag{4.4.15}$$

This is exactly (4.3.11). Therefore, (4.3.12),(4.3.13),(4.3.14), all hold. One concludes that for $0 \le t \le T$

$$\|u\|_{1,2}^4 \le \tilde{m}, \|u\|_\infty < \infty \tag{4.4.16}$$

By the same argument to prove Theorem 4.3.3 one has

THEOREM 4.4.1 (GLOBAL EXISTENCE THEOREM (II)). *For $u_0 \in H^2, Q \in C^2, \alpha\beta > 0$, there exists a unique global classical solution to (4.1.1).*

COROLLARY 4.4.2. *One could study half-line problem for the following GL equation by setting up $\gamma = \nu = \kappa = \epsilon, \alpha = \beta = \pm 1$ in (4.1.1):*

$$u_t = (\epsilon \pm i)u_{xx} - (\epsilon \pm i)|u|^2 u + \epsilon u \tag{4.4.17}$$

with initial-boundary data $u(x,0) = u_0(x) \in H^2[0,\infty), u(0,t) = Q(t) \in C^2[0,\infty)$. This is a perturbed NLS. Global existence was established in [18] for NLS ($\epsilon = 0$). It turns out that half-line problem for this particular perturbed NLS has global solution by Theorem 4.4.1. We shall indicate that if $u_\epsilon(x,t)$ solves (4.4.17), $u(x,t)$ solves the half-line NLS with the same initial-boundary data, then $u_\epsilon \to u(\epsilon \to 0)$ under certain norm.

REMARK 4.4.3. *The sign of $\alpha\beta$ has some significant implications in NLS case ($\nu = \kappa = \gamma = 0$). It was shown that if $\alpha\beta > 0$ there are no bound states (permanent waves) and the final state is just the similarity solution (cf.[1]). On the other hand, if $\alpha\beta < 0$ the x independent solutions of NLS are unstable (analogous to the Benjamin-Feir criterion [41]). If $\kappa < 0$, [41] indicated that the solution to (3.1.1) becomes unbounded in finite time. (This result is consistent with that found by Stuart and Stewartson (cf. [55]) who examined a Cauchy problem for the GL equation (3.1.1)). For the GL equation, the criterion $\nu\kappa + \alpha\beta > 0$ has physical implications (see [41] for details). Since $\nu, \kappa > 0$, our criterion $\alpha\beta > 0$ certainly implies $\nu\kappa + \alpha\beta > 0$. One has reason to believe that for certain initial-boundary data, global-existence can not hold for the half-line problem (4.1.1) if $\kappa < 0$, but an analytical proof may be very difficult because of the infinite domain. (There are not many blow-up results available: many successful attempts require a finite domain in order to estimate $\|u\|_4^4, \|u\|_6^6$ etc.) One should note that $\nu > 0$ is necessary to establish (4.1.6) for existence. Whether there is a blow-up for certain initial-boundary data in case $\alpha\beta < 0$ and $|\beta| > \sqrt{3}\kappa$ is still pending further investigation.*

Chapter 5.

WEAK SOLUTION TO AN INITIAL-BOUNDARY VALUE PROBLEM FOR THE GINZBURG-LANDAU EQUATION

§5.1 PRELIMINARIES.

The Cauchy problem for the GL equation (3.1.1) with $x \in \Omega = [0, L], u(x, 0) \in H_0^1(\Omega)$ is well-posed as can be seen by using classical techniques of nonlinear parabolic equations (cf.[2,27,38]). In this chapter we shall study the GL equation posed in the finite domain $\Omega = [0, L]$ with the boundary data $u(0, t) = Q(t), u(L, t) = 0$ and initial data $u(x, 0) = u_0(x)$. Under certain conditions on these data, we show that there is a unique weak solution. The solution u obtained here may not be a classical solution because $u_0 \in H^1$ only.

The concerned GL equation is posed on a bounded domain as follows ($\kappa, \nu > 0, \alpha, \beta$ real):

$$u_t = (\nu + i\alpha)u_{xx} - (\kappa + i\beta)|u|^2 u + \gamma u \tag{5.1.1}$$

$$x \in \Omega = [0, L], t \in [0, T], u(x, 0) = u_0(x), u(0, t) = Q(t), u(L, t) = 0$$

Here $Q(t), u_0(x)$ are complex functions. We shall, throughout this section, assume that for $t \in [0, T]$, $Q(t) \in C^2[0, \infty), Q(t) \neq 0, u_0(x) \in H^1(\Omega), u_0(L) = 0, u_0(0) = Q(0)$.

First we use the following transformation (here $w(x, t)$ is an appropriate smooth function satisfying $w(0, t) = 1, w(L, t) = 0$, which will be determined later)

$$u(x, t) = v(x, t) + Q(t)w(x, t) \tag{5.1.2}$$

and this substitution in (5.1.1) yields

$$v_t = (\nu + i\alpha)v_{xx} - (\kappa + i\beta)|v|^2 v + G_1 + G_2 + f \tag{5.1.3}$$

$$v(0,t) = v(L,t) = 0, v(x,0) = u_0(x) - Q(0)w(x,0)$$

$$f = -Qw_t - Q'w + \gamma Qw + (\nu + i\alpha)Qw_{xx} - (\kappa + i\beta)|Qw|^2 Qw \tag{5.1.4}$$

$$G_1 = c_1 v + c_2 \bar{v} = -(\kappa + i\beta)(2|Qw|^2 Qv + w^2 Q^2 \bar{v}) + \gamma v \tag{5.1.5}$$

$$G_2 = c_3 v^2 + c_4 |v|^2 = -(\kappa + i\beta)(\bar{w}\bar{Q}v^2 + 2wQ|v|^2) \tag{5.1.6}$$

LEMMA 5.1.1. *There exists* $w \in C^1(0,T;L^2(\Omega)) \cap C^0(0,T;H^2(\Omega))$, $w(0,t) \equiv 1, w(L,t) \equiv 0$ *such that* $f \in L^2(0,T;H_0^1(\Omega)), \partial_t f \in L^2([0,T] \times \Omega)$.

PROOF: Consider the following initial-boundary value problem:

$$w_t = (\nu + i\alpha)w_{xx} - \frac{L-x}{L}(\frac{Q'}{Q} + (\kappa + i\beta)|Q|^2 - \gamma) \tag{5.1.7}$$

$$w(0,t) = 1, w(L,t) = 0, w(x,0) = \frac{L-x}{L}$$

If we use the transformation $w(x,t) = W(x,t) + \frac{L-x}{L}$ then (5.1.7) is equivalent to

$$W_t = (\nu + i\alpha)W_{xx} - \frac{L-x}{L}(\frac{Q'}{Q} + (\kappa + i\beta)|Q|^2 - \gamma) = (\nu + i\alpha)W_{xx} + g \tag{5.1.8}$$

$$W(0,t) = W(L,t) = 0, W(x,0) = 0$$

Define $A = (\nu + i\alpha)D_x^2, D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ then clearly $D(A), H_0^1(\Omega)$ are dense in $L^2(\Omega)$ and A is closed. For $W \in H_0^1(\Omega)$,

$$(5.1.9)\qquad |((\lambda - A)W, W)| \geq |\lambda\|W\|_2^2 + \nu\|W'\|_2^2 + i\alpha\|W'\|_2^2| \geq c_0\|W\|^2_{H_0^1(\Omega)}$$

Thus $\lambda - A$ maps $D(A)$ 1-1 onto $L^2(\Omega)$. Also from (5.1.9) one has $|((\lambda - A)W, W)| \geq \lambda\|W\|_2^2$ thus $\|\lambda - A\| \leq \frac{1}{\lambda}$ for $\lambda > 0$. By Hille-Yosida Theorem, A generates a continuous contraction semigroup $N(t) = \exp At$ for $t > 0$. Since $Q \in C^2, Q \neq 0$, it is clear from (5.1.8) that $g'(t)$ is continuous. Thus $\int_0^t N(t-s)g(s)ds \in D(A)$ by [53] hence there is a unique local solution W to (5.1.8) with $W \in C^1(0, T_M; L^2(\Omega)) \cap C^0(0, T_M; D(A))$:

$$(5.1.10)\qquad W(t) = N(t)W_0 - \int_0^t N(t-s)\frac{L-x}{L}(\frac{Q'}{Q} + (\kappa + i\beta)|Q|^2 - \gamma)ds$$

It is not difficult to show that $T_M = T$. By using the fact $N(t)$ is a contraction semigroup in $L^2(\Omega)$ one finds through (5.1.10) that $\|W\|_2 < \infty$ on $[0, T]$. By standard estimate on W_t (cf.[53]) and the fact $Q \in C^2$ one finds that $\|W_t\|_2 < \infty$ on $[0, T]$. Thus $\|W\|_{D(A)} < \infty$ through (5.1.1). Consequently $w \in C^1(0, T; L^2(\Omega)) \cap C^0(0, T; H^2(\Omega))$ is the unique solution to (5.1.7).

It remains to check that $f \in L^2(0, T; H_0^1(\Omega)), \partial_t f \in L^2([0, T] \times \Omega)$. One looks at (5.1.4) and notes that when w solves (5.1.7):

$$(5.1.11)\qquad f = Q(-w_t + (\nu + i\alpha)w_{xx}) - (\kappa + i\beta)|Qw|^2 Qw + \gamma Qw - Q'w$$

$$= Q\frac{L-x}{L}(\frac{Q'}{Q} + (\kappa + i\beta)|Q|^2 - \gamma) - (\kappa + i\beta)|Qw|^2 Qw + \gamma Qw - Q'w$$

Evidently $f(0, t) = f(L, t) = 0$. Also, $w \in H^2(\Omega) \Rightarrow |w|^2 w \in H^1(\Omega)$, thus $f \in L^2(0, T; H_0^1(\Omega))$. Since $Q \in C^2$ and $w, w_t \in C^0(0, T; L^2(\Omega)), \partial_t f \in L^2([0, T] \times \Omega)$ is obvious. Q.E.D.

§5.2 EXISTENCE OF A WEAK SOLUTION.

We shall utilize Galerkin's method (cf.[12, 38]) to solve (5.1.3). Let

$$w_j \in H_0^1(\Omega), -\Delta w_j = \lambda_j w_j, j = 1, 2, ... \tag{5.2.1}$$

and $v_m = \sum_1^m g_{jm}(t) w_j$ be an approximate solution where $\{g_{jm}\}_{j=1}^m$ are determined by the conditions

$$(\partial_t v_m, w_j) + (\nu + i\alpha)(\partial_x v_m, \partial_x w_j) + (\kappa + i\beta)(|v_m|^2 v_m, w_j) \tag{5.2.2}$$

$$= (G_1 + G_2 + f, w_j)$$

$$v_m(0) = v_{0m} \in [w_1, ..., w_m], v_{0m} \to v_0 = v(x, 0) \in H_0^1(\Omega)$$

Multiply (5.2.2) by $\bar{g}_{jm}(t)$ and add them for $j = 1, 2, ..., m$. Now one has

$$(\partial_t v_m, v_m) + (\nu + i\alpha)\|\partial_x v_m\|_2^2 + (\kappa + i\beta)\|v_m\|_4^4 = (G_1 + G_2 + f, v_m) \tag{5.2.3}$$

From (5.1.5),(5.1.6) it is clear that $|c_i(x,t)| \le c_0, i = 1, 2, 3, 4$. Take real part of (5.2.3):

$$\frac{1}{2}\partial_t \|v_m\|_2^2 + \nu\|\partial_x v_m\|_2^2 + \kappa\|v_m\|_4^4 \tag{5.2.4}$$

$$\le 2c_0\|v_m\|_2^2 + 2c_0 \int_\Omega |v_m|^3 dx + \|f\|_2 \|v_m\|_2$$

$$\le 2c_0\|v_m\|_2^2 + \kappa\|v_m\|_4^4 + \frac{1}{\kappa}c_0^2\|v_m\|_2^2 + \|f\|_2^2 + \frac{1}{4}\|v_m\|_2^2 = \kappa\|v_m\|_4^4 + c'\|v_m\|_2^2 + \|f\|_2^2$$

Since $\nu > 0$,(5.2.4) implies that $\partial_t\|v_m\|_2^2 \le 2c'\|v_m\|_2^2 + 2\|f\|_2^2$ and by Gronwall lemma one has $\|v_m\|_2^2 \le M$ on $[0, T]$. By (5.2.1), one could replace w_j by Δw_j in (5.2.2) to get

$$(5.2.5) \qquad (\partial_t v_m, \Delta v_m) + (\nu + i\alpha)(\Delta v_m, \Delta v_m) - (\kappa + i\beta)(|v_m|^2 v_m, \Delta v_m)$$

$$= -(G_1 + G_2 + f, \Delta v_m)$$

$$(5.2.6) \qquad \frac{1}{2}(\int_0^t |Q(\tau)|^2 d\tau)^{\frac{1}{2}} \|\partial_x v_m\|_2^2 + \nu \|\Delta v_m\|_2^2$$

$$\leq (\kappa + |\beta|)\|v_m\|_6^3 \|\Delta v_m\|_2 + \|G_1 + G_2 + f\|_2 \|\Delta v_m\|_2$$

$$\leq \frac{\nu}{2}\|\Delta v_m\|_2^2 + \frac{(\kappa + |\beta|)^2}{2\nu}\|v_m\|_6^6 + \frac{\nu}{2}\|\Delta v_m\|_2^2 + \frac{1}{2\nu}\|G_0 + G_1 + f\|_2^2$$

Now use $\|v_m\|_2^2 \leq M$ and the following Gagliardo-Nirenberg estimates (cf.[50]):

$$(5.2.7) \qquad \|v_m\|_6 \leq m\|\partial_x v_m\|_2^{\frac{1}{3}} \|v_m\|_2^{\frac{2}{3}} + m'\|v_m\|_2, \|v_m\|_4$$

$$\leq m_0 \|\partial_x v_m\|_2^{\frac{1}{4}} \|v_m\|_2^{\frac{3}{4}} + \hat{m}\|v_m\|_2$$

Then (5.2.6) becomes

$$(5.2.8) \qquad \partial_t \|\partial_x v_m\|_2^2 \leq \tilde{c}\|v_m\|_6^6 + \hat{c}(\|v_m\|_2 + \|v_m\|_4^2 + \|f\|_2)^2$$

$$\leq c + \bar{c}\|(\int_0^t |P(\tau)|^2 d\tau)^{\frac{1}{2}} v_m\|_2^2$$

Therefore, v_m is bounded in $L^\infty(0, T; H_0^1(\Omega))$. Now differentiate (5.2.2) with respect to t:

$$(5.2.9) \qquad (\partial_t^2 v_m, w_j) + (\nu + i\alpha)(\partial_t \partial_x v_m, \partial_x w_j)$$

$$+(\kappa + i\beta)(2|v_m|^2 \partial_t v_m + v_m^2 (\int_0^t |Q(\tau)|^2 d\tau)^{\frac{1}{2}} \bar{v}_m, w_j)$$

$$= (\partial_t G_1 + \partial_t G_2 + \partial_t f, w_j)$$

One could replace w_j by $\partial_t v_m$ in (5.2.9) and take the real part:

$$(5.2.10) \qquad \frac{1}{2}\partial_t\|\partial_t v_m\|_2^2 + \nu\|\partial_t\partial_x v_m\|_2^2$$

$$\leq |(\kappa+i\beta)(2|v_m|^2\partial_t v_m + v_m^2(\int_0^t |Q(\tau)|^2 d\tau)^{\frac{1}{2}}\bar{v}_m, \partial_t v_m)|$$

$$+|(\partial_t G_1 + \partial_t G_2 + \partial_t f, \partial_t v_m)|$$

$$\leq c' M_0^2\|\partial_t v_m\|_2^2 + c_0\|\partial_t v_m\|_2 + \bar{c}\|\partial_t v_m\|_2^2 \leq C_0 + C\|\partial_t v_m\|_2^2$$

Here we use (5.1.5), (5.1.6) and $\|v_m\|_\infty \leq \bar{M}\sqrt{\|\partial_x v_m\|_2\|v_m\|_2} \leq M_0$. By [38], $\partial_t v_m$ is bounded in $L^\infty(0,T;L^2(\Omega))$ and we can extract v_μ from v_m so that $v_\mu \to v$ weakly in $L^2(0,T;H_0^1(\Omega)); \partial_t v_\mu \to \chi$ weakly in $L^2(0,T;L^2(\Omega)), \partial_t v = \chi$ and v solves

$$(5.2.11) \qquad (\partial_t v, h) + (\nu+i\alpha)(\partial_x v, \partial_x h) + (\kappa+i\beta)(|v|^2 v, h) = (G_1 + G_2 + f, h)$$

$\forall h \in H_0^1(\Omega)$. We deduce from (5.1.2),(5.1.3) and (5.2.11) that $u = v + Qw$ satisfies

$$(5.2.12) \qquad (\partial_t u, h) + (\nu+i\alpha)(\partial_x u, \partial_x h) + (\kappa+i\beta)(|u|^2 u, h) - \gamma(u,h) = 0$$

$\forall h \in H_0^1(\Omega)$ with $u \in L^\infty(0,T;H^1(\Omega)), \partial_t u \in L^\infty(0,T;L^2(\Omega))$ and $u(x,0) = u_0(x), u(0,t) = Q(t), u(L,t) = 0$.

To show the uniquess of solution u, it suffices to show the uniqueness of solution v to (5.2.11). Suppose v_1, v_2 are two solutions and write $V = v_1 - v_2$. By (5.2.11),

$$(5.2.13) \qquad (\partial_t V, h) + (\nu+i\alpha)(\partial_x V, \partial_x h) + (\kappa+i\beta)(|v_1|^2 v_1 - |v_2|^2 v_2, h)$$

$$= (G_1(v_1) - G_1(v_2) + G_2(v_1) - G_2(v_2), h), \forall h \in H_0^1(\Omega)$$

In particular, we can replace h by V in (5.2.13) then take the real part:

$$(5.2.14) \qquad \frac{1}{2}\left(\int_0^t |Q(\tau)|^2 d\tau\right)^{\frac{1}{2}} \|V\|_2^2 + \nu\|\partial_x V\|_2^2 \le c_0 \||v_1|^2 v_1 - |v_2|^2 v_2\|_2 \|V\|_2$$

$$+\|G_1(v_1) - G_1(v_2)\|_2 \|V\|_2 + \|G_2(v_1) - G_2(v_2)\|_2 \|V\|_2 \le c\|V\|_2^2$$

Here we use $v_1 = V + v_2, \|v_i\|_{\frac{1}{4}} \le c'$ for $i = 1, 2, \|V\|_\infty \le 2c'$ and $c = c(c', Q, w, \alpha, \beta, \kappa, \nu, \gamma, T)$. Since $\nu > 0$, $V \in H_0^1(\Omega)$, (5.2.14) becomes $\|V\|_2^2 \le 2c\int_0^t \|V\|_2^2 d\tau$ thus $V \equiv 0$ on $[0, T]$ because c is independent of t. We have therefore completed the proof of the following result:

THEOREM 5.2.1 (WEAK SOLUTION). *For the GL equation (5.1.1) there exists a unique weak solution u satisfying (5.2.12).*

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