TECHNICAL NOTES

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No. 126.

ABSOLUTE DIMENSIONS OF KARMAN VORTEX MOTION.

By Werner Heisenberg.

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ABSOLUTE DIMENSIONS OF KARMAN VORTEX MOTION.*

By Werner Heisenberg.

Professor Karman succeeded in calculating the resistance W of a plate, moving perpendicularly to its surface through water with the velocity U, from data obtained directly from the phenomenon of the flow.**

Professor Karman investigated the flow for some distance behind the plate. The experiment showed a regular arrangement of vortex lines, a "vortex street" (Fig. 2), which, remaining behind the plate, advanced more slowly than the latter. The relative dimensions of this arrangement, i.e. the ratio of the distance l between the vortices to the width h of the vortex street (Fig. 1), were determined on the basis of a stability investigation. The velocity u and a linear dimension of the vortex system (for instance, the distance l between two successive vortices rotating in the same direction) had to be found experimentally, in order that the resistance W could actually be deduced. The question as to the origin of the vortex system remained unanswered.

According to the law of Helmholtz that no vortex can form in a frictionless or non-viscous fluid, the viscosity is obviously responsible for the formation of the vortices. Karman also

* From Physikalische Zeitschrift, September 15, 1922, pp. 363-366.

**Karman, Göttinger Nachr. 1911, p. 509, and 1912, p. 547; Karman and Rubach, Physikalische Zeitschrift, 1912, p. 49.

considered that an investigation into the phenomena of the boundary layer at the plate would be necessary, in order to calculate the unknown u and l of the vortex tail. According to the calculations of Oseen (Ann. d. Phys. 1915, pp. 231 and 646), we must nevertheless assume that the influence of viscosity in immediate proximity to the plate is less than at some distance behind it. Furthermore, it was shown by Jaffé (Phys. Zeitschr. 1920, p. 541), that, even in a viscous fluid, in general, no vortex can originate and that therefore the reason for the formation of vortices appearing everywhere in hydrodynamics cannot be found in the viscosity. According to Jaffé, there is much more cause for the formation of vortices, when there are discontinuities in the external forces or in the velocity of the fluid. In the case under consideration, there are certainly such discontinuities present in th vicinity of the plate. We will therefore attempt to determine the quantities u and l, respectively $\frac{u}{U}$ and $\frac{l}{d}$ (d = width of plate from a consideration of these discontinuities.

The only* hitherto known form of flow of water past flat plates, when the flow velocity past the edge of the plate is not infinite, is the Helmholtz-Kirchhoff unstable potential motion (Fig. 1).**

The practical impossibility of this flow, which, on account of its surfaces of discontinuity, must be regarded as a vortex

* Prof. Jaffé has kindly reminded me that this is not exactly accurate. A. R. Richardson (Phil. Mag. 1919, p. 433) offers another very interesting mathematical solution, which, however, can hardly have any physical significance.

** Lamb, Lehrbuch der Hydrodynamik, IV, Sec. 76, p. 113.

flow, lies likewise in the lack of stability of the vortex sheets. There is always the conjecture that the unstable potential motion must exist at least in the immediate vicinity of the plate.

In fact, photographs show that the actual flow in the immediate vicinity of the plate is remarkably similar to the Kirchhoff potential motion. In connection with the origin of the Karman vortex system, we will therefore investigate the following hypothesis (Fig. 1)*.

In the immediate vicinity of the plate there is first formed the unstable potential motion. At some distance from the plate, the surfaces of discontinuity, i.e. the vortex layers of this potential motion, on account of their lack of stability, roll up and form the Karman vortex arrangement at a still greater distance from the plate.

This hypothesis can approach reality, only in so far as the vortices, originating at some distance, influence the flow in the immediate vicinity of the plate. Furthermore, it can hold good only for the front side and edges of the plate, since, in the "dead water," there is always a very indefinite vortex motion (but not vortex formation).

However simple and plausible the thus formulated statement may at first seem, serious objections may be raised against it, as Professor Prandtl has very kindly informed me. (See his remarks at the close of this article). Professor Karman put these objec-

* In Fig. 1 the velocities ($w = u - i v$) are in part introduced in double fashion. The bracketed data refer to velocities with relation to the plate, while the other data refer to velocities with reference to the fluid at infinity.

tions in the following form: "If the statement is applied literally to the Bobileff case of a symmetrical wedge, it gives, for all the angles of the wedge, the same drag coefficient, (namely, the same as for the flat plate)." The question remains open, therefore, as to why, just in our case, the statement corresponds so closely to the fact. In any case, further calculation will show that, for the plate, the above hypothesis agrees remarkably well with experience.

1. Condition for the Velocity of the Vortex System.

For calculating the unknown u and l, two equations are readily found. It follows at once from the law of the conservation of the vortex moment, which we apply to both vortex sheets, that the vortex moment originating per unit of time at each of the two edges of the plate must equal the vortex moment, of either direction of rotation, carried away per unit of time by the vortex system. This vortex moment has the value

$$\frac{\zeta}{l}(U - u) \qquad (1)$$

when ζ stands for the moment of the individual vortex filament. On the other hand, in order to calculate the vortex moment originating on the edge of the plate, we note that, in the Helmholtz unstable potential motion, there is in the "dead water" behind the plate (considered stationary) the velocity 0, while in the "free" surface proceeding from the plate it is U, and hence the discontinuity of the velocity is in the instability surface U (Fig. 1). If we separate the instability surface into its elements $d f$, which we consider as vortex lines, and form the velocity integral around a closed circuit enclosing such a vortex line d f (Fig. 1), we find its moment to be U d f. Moreover, this element d f is not at rest with reference to the plate, since we would then have the velocity -U/2 in the "dead water" and U/2 outside of it. The element moreover moves farther from the plate on the instability surface with the velocity U/2. Hence we conclude that the vortex moment originating per unit of time at the edge of the plate is

$$U \frac{U}{2} \qquad (2)$$

From Equations (1) and (2) we obtain

$$U \frac{U}{2} = \frac{\zeta}{l} (U - u) \qquad (3)$$

3. Condition for the Vortex Interval.

A second relation follows from the fact that there is always a certain amount of water (Fig. 2), which is driven forward between the two series of vortices. While the fluid outside the "vortex street" on the average is at rest (i.e. any fluid particle does not depart in the course of time any great distance from its original position), a stream flows continuously forward within the "street." The quantity of water carried along in this stream must equal the quantity constantly pushed ahead by the plate in its motion, which is Ud.

In order to calculate the former, we write, following Karman, the complex potential of the vortex flow

$$X = \frac{i\zeta}{2\pi} \lg \frac{\sin (z_o - z)\frac{\pi}{l}}{\sin (z_o + z)\frac{\pi}{l}} \tag{4}$$

$$z_o = \frac{l}{4} + \frac{hi}{2}$$

in which h denotes the distance between the two series of vortices.

$z = x + i\,y$ (The X-axis runs, as shown in Fig. 1, parallel to the vortex series and divides the space evenly between them). In order to obtain the desired quantity of water, we integrate, along an arbitrary path of line elements $d\,s$ ($\overline{u}$ = velocity vector; $w = u - i\,v$; J denotes "Imaginary portion of"),

$$\int_a^b \overline{u}_n\, ds = \int_a^b J\,(wdz) = J \int_a^b \frac{dX}{dz}\, dz = J\,(X)\Big|_a^b$$

The limits a and b of the integral are the two streamlines bounding the desired mass of water (dash lines in Fig. 2), in which the particular position of the points a and b on the stream lines is of no significance. In the case under consideration, the limiting stream lines are the only ones (in the Y-direction) which extend to infinity. We therefore extend the integral simply from $z = + i\infty$ to $z = - i\infty$. In Fig. 2, the integration path from $- i\infty$ extends along a dash line near the vortex, then transversely across the stream bed and thence again along the limiting stream line to $+ i\infty$. The quantity of fluid passing through the stream per unit of time is accordingly

$$J\,(X_{+i\infty} - X_{-i\infty}).$$

But according to Equation (4)

$$X_{\alpha+\beta i} = \frac{i\zeta}{2\pi} lg \frac{e^{i\frac{\pi}{l}\left(\frac{l}{4}+\frac{hi}{2}-\alpha-\beta i\right)} - e^{-i\frac{\pi}{l}\left(\frac{l}{4}+\frac{hi}{2}-\alpha-\beta i\right)}}{e^{i\frac{\pi}{l}\left(\frac{l}{4}+\frac{hi}{2}+\alpha+\beta i\right)} - e^{-i\frac{\pi}{l}\left(\frac{l}{4}+\frac{hi}{2}+\alpha+\beta i\right)}}$$

Hence, for $\beta \to \infty$, $\alpha = 0$ (α does not enter into the imaginary portion of X),

$$X_{+i\infty} = \frac{i\zeta}{2\pi} lg\left(-e^{+\frac{i\pi}{2}-\frac{h}{l}\cdot\pi}\right) = -\frac{3\zeta}{4} - \frac{i\zeta h}{2l}$$

$$X_{-i\infty} = \frac{i\zeta}{2\pi} lg\left(-e^{-\frac{i\pi}{2}+\frac{h}{l}\pi}\right) = -\frac{\zeta}{4} + \frac{i\zeta h}{2l}$$

Therefore the quantity of fluid (no attention being paid to the signs, since only the absolute value concerns us) is

$$\left|J\ (X_{+i\infty} - X_{-i\infty})\right| = \frac{\zeta \cdot h}{l} \tag{5}$$

from which is obtained the second relation

$$U\ d = \frac{\zeta \cdot h}{l} \tag{6}$$

3. Conclusion.

For the numerical evaluation of Equations (3) and (6), we utilize Karman's formulas (l. c.)

$$\frac{\zeta}{l\sqrt{8}} = u \tag{7}$$

$$\text{and}\quad \frac{h}{l} = 0.283 \tag{8}$$

From Equation (3) then follows

$$\tfrac{1}{2}U^2 = u\sqrt{8}\,(U - u),$$

$$1 = 2\sqrt{8}\,\frac{u}{U}\left(1 - \frac{u}{U}\right),$$

$$\frac{u}{U} = 0.229. \qquad (9)$$

The other root $u/V = 0.771$ is excluded, for a reason about to be given. Equation (6), on account of Equations (7) and (8), is converted into

$$U \cdot d = u \cdot \sqrt{8} \cdot l \cdot 0.283.$$

Whence there follows, according to Equation (9)

$$\frac{l}{d} = \frac{1}{\sqrt{8} \cdot 0.283 \cdot 0.229} = 5.45; \qquad (10)$$

$$\frac{h}{d} = 1.54 \qquad (11)$$

The other root for u/V would give a much smaller value for l/d and h/d. Since, however, the discontinuity surfaces of the potential flow are directed outward, it is quite obvious that this second possibility is unattainable.

Karman gives, as empirical values,

$$\frac{u}{U} = 0.20; \quad \frac{l}{d} = 5.5.$$

From the theoretical values for u/V and l/d we have $\psi_w = 0.90$ as the value for the specific coefficient of drag ψ_w, defined by

$$W = \psi_w \cdot L \cdot d \cdot U^2 \cdot \rho$$

in which L = length of plate at right angles to the plane of the diagram and ρ = density of fluid.

This value seems to agree well with the latest determinations.

Institute for Theoretical Physics,

Munich, July 18, 1922.

Remarks by Prof. L. Prandtl on the Foregoing Article.

My objections to the statements of Mr. Heisenberg, already referred to by him, are briefly formulated in the following paragraph.

1. The resistance of the plate is about twice as great as given by the Kirchhoff formula. Behind the plate there are only small velocities and hence a constant pressure clear to the edges. To the fall in pressure from the middle of the front side to the edge, which is about twice as great, there also corresponds a value of $U^2/2$ about twice as great. Hence the vortex production per second must be $\alpha\, U^2/2$, in which α is about 2 for the flat plate. (To be exact, the motion back of the plate also exerts a slight influence on the vortex production).

2. It is not impossible that appreciable portions of the positive and negative vortex moments mutually eliminate one another by intermixing in the turbulent zone behind the plate and hence are no longer present in the vortex system as it flows

away. If the eliminated fraction is β, we would have, in place of Equation (3)

$$(1 - \beta)\,\alpha\,\frac{U^2}{2} = \frac{\zeta}{l}\,(U - u)$$

3. The conclusion, that the quantity of fluid flowing between the vortices is exactly Ud, is not convincing. The limit of the vortex zone advances with the plate into the fluid (regarded as at rest) and this place must also be filled by the flow. An opposite effect may be introduced by the contraction of the vortex zone in forming the vortex series, so that Equation (6) also has an undetermined factor, which may be greater or less than unity.

In my opinion, Mr. Heisenberg's computation, though very instructive, is only adapted to yield definite conclusions when used in connection with experimental data concerning the correction factors referred to above.

Göttingen, July 29, 1922.

Translated by the National Advisory Committee for Aeronautics.

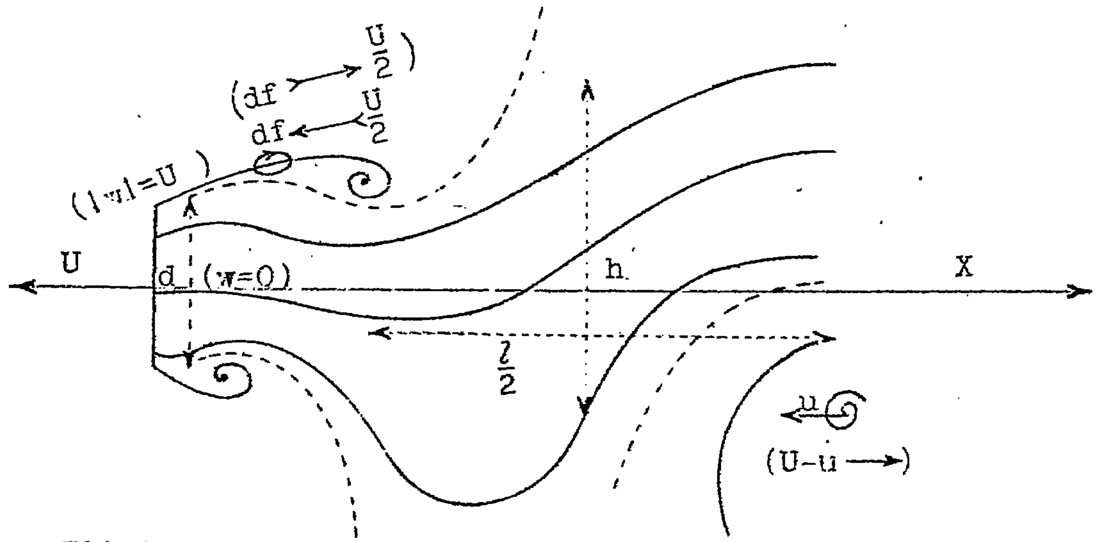

Fig.1

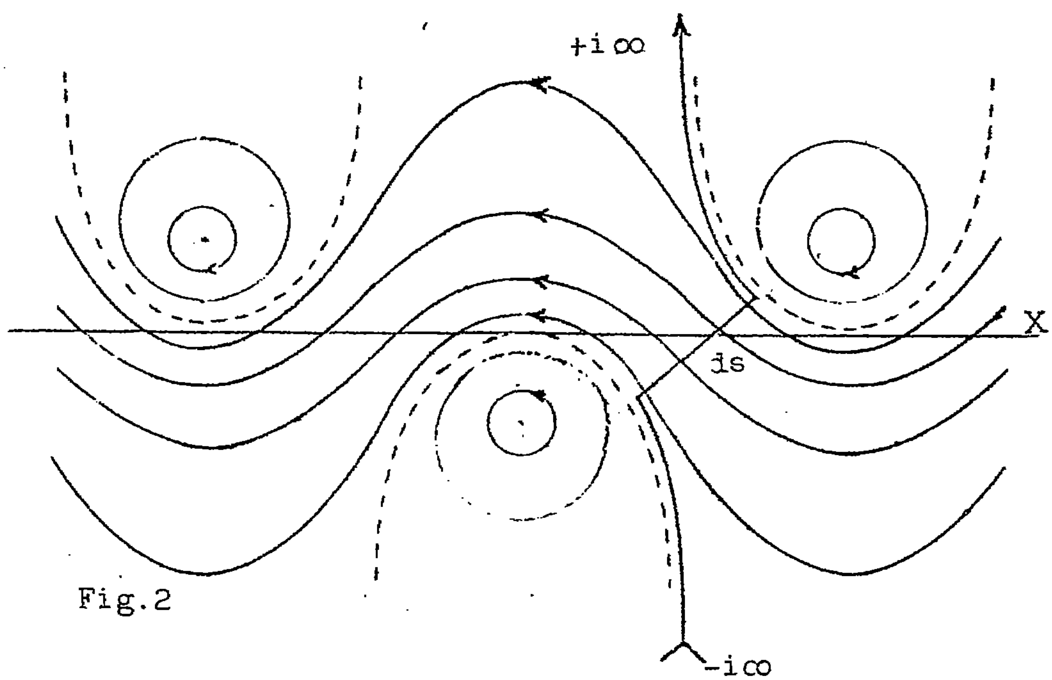

Fig.2