A General Coupled Morning-evening Traffic Equilibrium Model with Rideshare, Ride-hailing, and Public Transit Services

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Abstract

We develop a general equilibrium model to capture the complex interactions between different modes, such as solo driving, public transit, as well as rideshare and ride-hailing services such as Uber and Lyft, under a joint morning and evening commute framework. Formulated as a variational inequality (VI) and equivalently as a mixed complementarity problem (MiCP), the model allows (a) travelers to switch between different transportation modes and (b) passengers from different Origin-Destination (OD) pairs to share a ride together. The computational results on the Sioux-Falls network show that our model captures the possible mode switches and the coupling effects between morning and evening commutes. Furthermore, our numerical examples demonstrate that modeling morning and evening commutes separately tends to overestimate the travelers’ disutility and the average Vehicle Miles Traveled (VMT) in the network.

Key words: coupled morning-evening traffic equilibrium, variational inequality, flexible rideshare, ride-hailing, public transit.
1 Introduction

App-based transportation services, such as ride-hailing services (also called e-hailing or ride-sourcing services in the literature, e.g., Ban et al., 2019; Xu et al., 2021) provided by Uber, Lyft, DiDi, Grab and Ola or casual rideshare enabled by Scoop, Waze, BlaBlaCar and Didi Hitch are growing rapidly. For example, Uber has hit its milestone in 2018 to serve over 10 billion trips within more than 700 cities of 80 countries (Uber, 2018). There are over 75 million riders and 3.9 million drivers in total, producing more than 14.1 billion dollars of annual net revenue (Iqbal, 2020). These emerging transportation services are transforming the travel behavior of individuals and urban mobility patterns, and provide significant challenges to transportation planners and policy makers on how to assess the impact of these services on transportation systems, and how to facilitate or regulate these services.

Due to heavy traffic, commuters suffer from long travel delays in both the morning and evening commutes in many urban areas. Transportation planners should consider both commuting trips in their analysis, since the travel choices in one commute affect those of the other, but in practice they are rarely jointly analyzed. With the emerging transportation services, one challenge for transportation planners is to quantify travelers’ possible mode switches between the morning commute and the evening commute. Empirical studies such as Mao et al. (2018) and Bachir et al. (2019) show that nearly 20% of travelers switch their travel modes between morning and evening commutes. The ride-hailing and rideshare services provide more travel mode choices for commuters in both morning and evening commutes. For example, a person can choose a rideshare service in the morning, but use a ride-hailing service for the evening return trip to reduce the pairing cost, and provide more flexibility in evening trips. This capturing of mode switches is especially important if the travel cost data is different in the morning and evening times. For example, a traveler with a high inconvenience cost for rideshare in the evening, which may be due to the need to pick up their children from after-school activities, will not use this mode in the evening. Thus, an alternative option for these travelers is to use rideshare in the morning and to take ride-hailing in the evening. Travelers’ mode switches between morning and evening commutes have already been captured in some models of transportation research, e.g., the bottleneck model (Gonzales and Daganzo, 2013; Zhang et al., 2019; Zhao et al., 2021) and discrete choice model (Hossain et al., 2021). However, how to capture that in the traditional traffic equilibrium model (Sheffi, 1985; Patriksson, 2015; Brederode et al., 2019) still remains open.

Another challenge for transportation planners is to capture the interactions between the various modes of transportation in the morning and evening commutes. With rideshare and ride-hailing services, the choice of travelers in the morning/evening may influence that in the evening/morning. For example, a traveler may decide to drive in the morning if (s)he knows that it is very expensive or inconvenient to take a ride or use public transit in the evening, which may be caused by the decrease of the evening supply in the ride-hailing markets or the lack of accessibility to public transit. There is a clear need to develop a coupled morning-evening modeling framework in supporting transportation planners’ decision-making.

In recent years, researchers have included rideshare services in the traditional traffic assignment problem. Xu et al. (2015a) and Xu et al. (2015b) first proposed the traffic equilibrium models with rideshare services. Considering an Origin-Destination (OD) based surge pricing strategy, Ma et al. (2020) modeled a rideshare user equilibrium with ride-matching constraints. Li et al. (2020) studied a path-based rideshare equilibrium model to simultaneously produce route choices, mode

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1In this paper, rideshare refers to the mode of carpooling with payments enabled by companies that distinguishes from traditional carpooling where no payment for the ride is made. It also differs from ride-hailing in terms of price and inconvenience cost.
choices, and matching decisions. Instead of using a mixed complementary formulation, Wang et al. (2021) established a convex programming formulation for the rideshare user equilibrium problem. Noruzoliaee and Zou (2022) formulated a rideshare user equilibrium model in an autonomous vehicle context. Some papers also extended the traditional traffic assignment problem by considering ride-hailing services. Ban et al. (2019) first modeled the ride-hailing services in a general equilibrium model. Li et al. (2021) proposed a network equilibrium model with optimal spatial pricing for ride-hailing services. To better understand vacant trips generated by ride-hailing services, Xu et al. (2021) put forward a network equilibrium model to capture both cruising and deadheading trips of ride-hailing vehicles. Chen and Di (2023) formulated a ride-hailing network equilibrium model considering pooling options for passengers. Di and Ban (2019) proposed a traffic equilibrium model which includes both rideshare and ride-hailing services. In order to capture the interactions between ride-hailing and public transit services together with traffic equilibrium, Wei et al. (2022) modeled the competition between ride-hailing service and public transit planning, while Ke et al. (2021) and Liu et al. (2023) studied the complementary and substituting relationship between these two services.

To the best of our knowledge, there is no research to provide a general equilibrium model to capture the complex interactions between solo driving, rideshare, ride-hailing, and public transit in a coupled morning-evening commute framework. There are several reasons for developing a coupled morning-evening traffic equilibrium model to assist transportation planners in their decision making, especially considering the emerging rideshare and ride-hailing services: First, even for the same transportation network, traffic equilibria in the morning and evening commutes are not symmetrical due to different road networks for the morning and evening trips. Asymmetrical cost structures for the morning and evening commutes could further enlarge this difference; Second, with the competition or cooperation between various transportation modes, travelers may choose one type of commute mode in the morning period, and switch to a different type in the evening period, especially when the cost structures differ between the morning and the evening; Third, with the flexibility provided by rideshare and ride-hailing services, the morning and evening commutes interact with each other since travelers’ choice in the morning may influence that in the evening and vice versa.

Although there are some papers to extend the bottleneck model (Vickrey, 1969; Li et al., 2020) as a Morning-evening Commute Problem (Zhang et al., 2008; Daganzo, 2013; Gonzales and Daganzo, 2013; Zhang et al., 2019; Zhao et al., 2021), for these papers the reason for considering both morning and evening commutes together is markedly different. The motivation for the bottleneck model is that the schedule penalty functions in the morning and evening vary. Finally, rideshare, ride-hailing, and public transit services have not been considered simultaneously in the Morning-evening Commute Problem.

In order to close the identified research gap for the traffic assignment problem, we propose a general equilibrium model that considers the joint travel decisions and interactions between solo driving, rideshare, ride-hailing, and public transit in a coupled morning-evening commute modeling framework. The main contributions of this paper are listed as follows:

- We develop a general equilibrium modeling framework to capture both rideshare and ride-hailing services as well as their interactions with traditional transportation modes such as solo driving and public transit between morning and evening commutes with the features of (i) providing simultaneously the results of traffic flows and travelers’ mode choices; (ii) quantifying the possible mode switches across various transportation services between the morning and evening commutes; (iii) allowing for passengers from different OD pairs to share a ride together; and (iv) capturing the coupling interaction effects between morning and evening commutes.
The proposed model is formulated as a variational inequality (VI), and reformulated as an equivalent mixed complementarity problem (MiCP). Then our proposed model is validated using the Sioux-Falls network. The experimental analysis shows the coupling effects on average Vehicle Miles Traveled (VMT), number of travelers to use each mode, and number of mode switches as a function of the cost parameters such as rideshare inconvenience. We show that a decoupled modeling approach will overestimate travelers’ disutility and average VMT.

The remainder of this paper is organized as follows. In Section 2, we provide the coupled morning-evening traffic equilibrium model with solo driving, rideshare, ride-hailing, and public transit. In Section 3, experimental results are given to illustrate our model. Section 4 concludes this paper and points out some possible directions for future research.

2 Mathematical Model

We propose an extended traffic equilibrium model of morning and evening commutes, taking into account the emergent travel trends of rideshare and ride-hailing that offer alternative modes of travel supplementing the traditional modes of commuting: solo driving and public transit. The goal of the model is to study the morning and evening commute trip flows in the network caused by traffic congestion and the travelers’ choices of commute types to minimize their disutilities. More importantly, our approach combines morning travel from an origin to a destination and evening return from the same destination (which therefore is the origin of the evening trip) to the morning’s origin; this round trip is composed of a morning trip taken on a path and an evening trip taken on a possibly different (reverse) path with possibly a different mode. The round-trip path flows and mode choices encompass travelers’ commute behavior; the equilibrium will determine the travelers’ path and mode selections by equilibrating the round-trip path flows, morning mode choices, and evening mode choices with the travelers’ disutilities based on an extension of Wardrop’s user equilibrium principle.

As illustrated in Fig. 1, there are different types of commuters: (1) drivers, labelled as d, including both solo and rideshare drivers labelled as sd and rd, respectively; namely d = {sd, rd}; (2) passengers, labelled as p, including rideshare, ride-hailing, and transit passengers labelled as rp, hp, and tp, respectively; thus p = {rp, hp, tp}; we use the letter t ∈ {sd, rd, rp, hp, tp} as the generic label for these 10 types of travelers (i.e., commuters). In the morning/evening commute, drivers can choose to provide rideshare services if it is convenient for them. But drivers will not provide ride-hailing services since they also have their own destinations. In this scenario, ride-hailing services are provided by another group of drivers who are not commuters. Part of the complication of the model is for rideshare drivers to pick up and drop off passengers, possibly involving some detours in terms of the driver’s more direct routes to and from work place. As a result, drivers can switch roles between solo driver and rideshare driver during morning/evening commute. For various reasons, passengers may switch from one type of commute mode in the morning to a different type in the evening: rideshare passengers, ride-hailing passengers, and transit passengers may switch among these three types.

Define $N_0$ as the set of all nodes in a network. A morning OD pair $k = (i_k; j_k)$ joining nodes $i_k$ and $j_k$ in $N_0$ becomes the OD pair $\bar{k} = (j_k; i_k)$ in the evening. That is to say, the origin and destination of morning OD pair $k \in K$ becomes the destination and origin of evening OD pair $\bar{k}$, respectively. Each traveler will choose a morning path $p^{am}_k \in P^{am}_k$ to go from home to work place and an evening path $p^{pm}_k \in P^{pm}_k$ to return home from work place. Mathematically, if $P_k$ denotes the set of all the paths throughout the entire day joining an OD pair $k$, then $P_k = P^{am}_k \times P^{pm}_k$. Let $P \triangleq \{P_k\}_{k \in K}$. Based on a set of travel costs, under a set of assumptions and subject to traffic congestion, the model
Figure 1. Mode choices and mode switches of morning and evening commutes.

aims to determine a user equilibrium of trips for all the paths \( p \in \mathcal{P} \) throughout the entire day. In this process, the model also determines the switches of the passenger types between the morning and the evening trips and also the switches among the drivers from solo to rideshare and vice versa.

2.1 Model notations and assumptions

Main notations used in this study are summarized in Table 1, 2, and 3, including input sets and parameters in Table 1, and decision variables in Table 2 and Table 3.

In order to balance model realism and mathematical tractability, we assume that:

- The modeling context is static. That is to say, this is a model from a planner’s perspective, instead of one at operational level.

- During the morning/evening commute, rideshare drivers may take a detour for picking up or dropping off rideshare passengers if needed. Thus, rideshare passengers with different OD pairs are allowed to share the same vehicle.

- A passenger does not change travel mode during his/her morning or evening trip. For example, during the morning commute, the passenger’s entire trip must be either completely a rideshare trip or completely a ride-hailing trip or completely a transit trip.

- If a traveler decides to be a driver in the morning commute, (s)he will drive the vehicle back home in the evening.

- Rideshare vehicles have the same passenger capacity, and each ride-hailing vehicle is assumed to pick up only one OD demand (passenger).

- The public transit service travels on specific infrastructure (e.g., tracks) and has its own right of way. Thus, it will not contribute to or be influenced by traffic congestion.

2.2 Constructing an extended network

To accommodate the switches between solo and rideshare drivers and the pick-up/dropoff locations, an extended network similar to the approach in Xu et al. (2015b) is constructed from the given network with splits of its node and arc sets. Specifically, each node of the original network is split to distinguish between drivers and passengers: a “driver arc” is split into a “solo-driver arc”
and a “rideshare-driver arc”; and a “passenger arc”, which is splitted into a “rideshare-passenger arc”, a “ride-hailing-passenger arc”, and a “transit-passenger arc”. The flows of solo-driver arcs, rideshare-driver arcs, and ride-hailing-passenger arcs will contribute to congestion. The totality of these splitted nodes and arcs defines the extended network.

Take Fig. 2 as an example, in Fig. 2(a), the original network consists of nodes $i, j, \ell$, and arcs $(i, j), (j, \ell)$. Thus, the extended network in Fig. 2(b) consists of:

- $N_0$: “driver” nodes $i, j$ and $\ell$;
- $N'_0$: “rideshare-passenger” nodes $i', j'$ and $\ell'$;
- $N''_0$: “ride-hailing-passenger” nodes $i'', j''$ and $\ell''$;
- $N'''_0$: “transit-passenger” nodes $i'''', j'''$ and $\ell'''$;
- $A_{sd}$: “solo-driver” arcs $(i, j)$ and $(j, \ell)$ (the one above in Fig. 2);
- $A_{rd}$: “rideshare-driver” arcs $(i, j)$ and $(j, \ell)$ (the one below in Fig. 2);
- $A_{rp}$: “rideshare-passenger” arcs $(i', j')$ and $(j', \ell')$;
- $A_{hp}$: “ride-hailing-passenger” arcs $(i'', j'')$ and $(j'', \ell''$);
- $A_{tp}$: “transit-passenger” arcs $(i''', j''')$ and $(j''', \ell''')$.

![Figure 2. The original network and the extended network.](image)

Note that these are not three disjoint networks since they are connected by the fixed demands, i.e., the sum of flows leaving the three splitted origin nodes (or entering the three splitted destination nodes) are fixed. As a result, travelers could choose to start at either node $i$ if they want to drive, node $i'$ if they wish to take the rideshare service, node $i''$ if they prefer using the ride-hailing service, or node $i'''$ if they would like to use the public transit service. From the extended network in Fig. 2(b), we can notice that if travelers choose to start from driver node $i$, they may travel on arc $a_{sd} \in A_{sd}$ and/or arc $a_{rd} \in A_{rd}$ before reaching the destination node $\ell \in N_0$, which is also a driver node. For example, in Fig. 2(b), a driver can be traveling along arc $(i, j) \in A_{rd}$ and then arc...
$(j, ℓ) \in A_{sd}$. That is to say, (s)he drives with some passenger(s) to share a ride from node $i$ to $j$, then drops off the passenger(s) at node $j$, and drives alone from $j$ to $ℓ$. However, once departing from a driver node, travelers cannot switch to passenger arcs in $A_{tp}$ or $A_{hp}$ or $A_{tp}$. The reason is here we have the assumption that drivers will not leave their vehicles until they arrive at their destinations. Observing that there is no arcs connecting a rideshare-passenger node $N_0'$ to a driver node $N_0$ or a ride-hailing-passenger node $N_0''$ or a transit-passenger node $N_0'''$. Thus, if a traveler departs from a rideshare-passenger node $i' \in N_0'$, (s)he is only allowed to travel on the rideshare passenger arcs $a_{tp} \in A_{tp}$ before (s)he reaches the destination node $ℓ' \in N_0'$. This is because we assume that travelers cannot change their travel mode before reaching their destinations once they choose to be rideshare passengers. Similarly, once travelers depart from a ride-hailing-passenger node $i'' \in N_0''$ (or a transit-passenger node $i''' \in N_0'''$), they are only allowed to travel on ride-hailing-passenger arcs $a_{hp} \in A_{hp}$ (or transit-passenger arcs $a_{tp} \in A_{tp}$) before they arrive at their destinations. Since $A_{sd} \cup A_{rd} \subset N_0 \times N_0$, $A_{tp} \subset N_0' \times N_0'$, $A_{hp} \subset N_0'' \times N_0'''$, and $A_{tp} \subset N_0''' \times N_0'''$, a path $p$ will either only visit nodes in $N_0$ using arcs in $A_{sd}$ or $A_{rd}$, or only visit nodes in $N_0'$ using arcs in $A_{tp}$, or only visit nodes in $N_0''$ using arcs in $A_{hp}$, or only visit nodes in $N_0'''$ using arcs in $A_{tp}$. Thus a path $p$ can contain only arcs of $A_{tp}$, or only arcs of $A_{hp}$, or only arcs of $A_{tp}$, or only arcs of $A_{sd} \cup A_{rd}$.

### 2.3 Congestion cost

Notice that the arc flows $f_{am}^a/\alpha_a$, $f_{am}^a/\alpha_a$, and $f_{am}^a/\alpha_a$, representing the number of solo driving vehicles, rideshare vehicles, and ride-hailing vehicles, respectively, are the sources of traffic congestion; but $f_{am}^a/\alpha_a$ and $f_{am}^a/\alpha_a$ are not (rideshare passenger flows and transit passenger flows do not influence traffic congestion). Let $f^a_{am} \triangleq \{f^a_{am}\}_{a \in A}$ and $f^a_{pm} \triangleq \{f^a_{pm}\}_{a \in A}$ where $A = A_{sd} \cup A_{rd} \cup A_{tp} \cup A_{hp} \cup A_{tp}$. Derived from the Bureau of Public Roads (BPR) function, we obtain the travel time $t_{fa}(t_a)$ as follows

$$t_{fa}^a(f^a_{am}) = t_a \left(1 + b \left(\frac{f^a_{am}}{\alpha_a} \right)^4\right) \quad \forall a \in A_0.$$

$$t_{fa}^a(f^a_{pm}) = t_a \left(1 + b \left(\frac{f^a_{pm}}{\alpha_a} \right)^4\right) \quad \forall a \in A_0.$$

where $a_{sd} = T_{sd}(a)$, $a_{rd} = T_{rd}(a)$, and $a_{hp} = T_{hp}(a)$ for $a \in A_0$ are the corresponding arcs for solo drivers, rideshare drivers, and ride-hailing vehicles respectively, splitted from the original arc $a \in A_0$; $t_a$ and $\alpha_a$ represent the free flow travel time and flow capacity of arc $a \in A_0$, respectively.
<table>
<thead>
<tr>
<th>Table 1. Input sets and parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Network structure</strong></td>
</tr>
<tr>
<td>$G_0 = (N_0, A_0)$</td>
</tr>
<tr>
<td>$K \subseteq N_0 \times N_0$</td>
</tr>
<tr>
<td>$\bar{k}$</td>
</tr>
<tr>
<td>$N_0'$</td>
</tr>
<tr>
<td>$N_{0''}$</td>
</tr>
<tr>
<td>$N_{0'''}$</td>
</tr>
<tr>
<td>$A_t$</td>
</tr>
<tr>
<td>$G = (N, A)$</td>
</tr>
<tr>
<td>$T_t(a_0)$</td>
</tr>
<tr>
<td>$T_0(a_t)$</td>
</tr>
<tr>
<td>$P_{km}$</td>
</tr>
<tr>
<td>$P_{km}$</td>
</tr>
<tr>
<td>$P_k$</td>
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<tr>
<td>$P$</td>
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<table>
<thead>
<tr>
<th><strong>Model parameters</strong></th>
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<tbody>
<tr>
<td>$D_k$</td>
</tr>
<tr>
<td>$M$</td>
</tr>
<tr>
<td>$\delta_{a,p}$</td>
</tr>
<tr>
<td>$tt_a(\bullet)$</td>
</tr>
<tr>
<td>$t_a, \alpha_a$</td>
</tr>
<tr>
<td>$r_{am/pm}^a$</td>
</tr>
<tr>
<td>$\psi$</td>
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<tr>
<td>$R_{am/pm}^a$</td>
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### Table 2. Primary and derived decision variables

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>( f_{am} )</td>
<td>Flow of travelers of arc ( a \in A ) in the morning</td>
</tr>
<tr>
<td>( f_{pm} )</td>
<td>Flow of travelers of arc ( a \in A ) in the evening</td>
</tr>
<tr>
<td>( z_p )</td>
<td>Flow of travelers of path ( p \in P ) through the entire day</td>
</tr>
<tr>
<td>( x_{am} )</td>
<td>Flow of travelers of arc ( a \in A ) and OD pair ( k \in K ) in the morning</td>
</tr>
<tr>
<td>( x_{pm} )</td>
<td>Flow of travelers of arc ( a \in A ) and reverse OD pair ( \bar{k} ) in the evening</td>
</tr>
<tr>
<td>( u_k )</td>
<td>Generalized (least) disutility of OD pair ( k \in K )</td>
</tr>
<tr>
<td>( \eta_{am}^{\pm} )</td>
<td>Morning shadow price of arc ( a \in A_0 ), induced by morning rideshare capacity constraint</td>
</tr>
<tr>
<td>( \eta_{pm}^{\pm} )</td>
<td>Evening shadow price of arc ( a \in A_0 ), induced by evening rideshare capacity constraint</td>
</tr>
<tr>
<td>( \mu_{i;am}^{k} )</td>
<td>Morning multiplier for OD pair ( k \in K ) and node ( i \in N ), induced by morning demand satisfaction constraint and morning flow conservation constraint</td>
</tr>
<tr>
<td>( \mu_{i;pm}^{\bar{k}} )</td>
<td>Evening multiplier for reverse OD pair ( \bar{k} ) and node ( i \in N ), induced by evening demand satisfaction constraint and evening flow conservation constraint</td>
</tr>
<tr>
<td>( \zeta_{a; k} )</td>
<td>Multiplier for OD pair ( k \in K ) and arc ( a \in A ), induced by driver flow conservation constraint</td>
</tr>
</tbody>
</table>

### Table 3. Cost functions

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I_{am}^{f_{am}} )</td>
<td>Morning inconvenience cost on arc ( a \in A_t ) experienced by commuter type ( t \in {rd, rp, hp, tp} ) as a function of morning arc flow ( f_{am} )</td>
</tr>
<tr>
<td>( I_{pm}^{f_{pm}} )</td>
<td>Evening inconvenience cost on arc ( a \in A_t ) experienced by commuter type ( t \in {rd, rp, hp, tp} ) as a function of evening arc flow ( f_{pm} )</td>
</tr>
<tr>
<td>( R_{am}^{f_{am}} )</td>
<td>Morning rideshare payment/income or ride-hailing payment on arc ( a \in A_t ) experienced by commuter type ( t \in {rd, rp, lp} )</td>
</tr>
<tr>
<td>( R_{pm}^{f_{pm}} )</td>
<td>Evening rideshare payment/income or ride-hailing payment on arc ( a \in A_t ) experienced by commuter type ( t \in {rd, rp, lp} )</td>
</tr>
<tr>
<td>( tt_{am}^{f_{am}} )</td>
<td>Morning travel time on arc ( a \in A_0 )</td>
</tr>
<tr>
<td>( tt_{pm}^{f_{pm}} )</td>
<td>Evening travel time on arc ( a \in A_0 )</td>
</tr>
<tr>
<td>( tc_{am}^{f_{am}} )</td>
<td>Total cost on arc ( a \in A ) experienced by morning commuters</td>
</tr>
<tr>
<td>( tc_{pm}^{f_{pm}} )</td>
<td>Total cost on arc ( a \in A ) experienced by evening commuters</td>
</tr>
<tr>
<td>( TC_p(z) )</td>
<td>Total cost on path ( p \in P ) experienced by commuters as a function of coupled morning-evening path flow ( z )</td>
</tr>
</tbody>
</table>

### 2.4 Inconvenience cost and payment/income

In this section, we define the inconvenience costs and payments/incomes of rideshare, ride-hailing and public transit services. The cost structure of rideshare is similar to that of existing rideshare user equilibrium literature (e.g., Xu et al., 2015b; Ma et al., 2020; Li et al., 2020; Ma et al., 2022). To include ride-hailing and public transit services in our general traffic equilibrium framework, we also formulate the inconvenience cost and payment for ride-hailing and transit passengers.

- **Inconvenience cost of rideshare drivers:** In addition to the congestion cost in Equation 1, a rideshare driver will also experience the inconvenience for sharing the vehicle with passengers, which
includes but is not limited to picking up, dropping off, or even waiting for passengers. The inconvenience cost of rideshare drivers is defined as follows

\[
I_{a;rd}^{am}(f_{a;rp}) \quad \forall a \in A_{rd} \text{ with } a_{rp} = T_{rp}(T_0(a))
\]  

(2)

where \(I_{a;rd}^{am}(\bullet)\) and \(I_{a;rd}^{pm}(\bullet)\) are monotone increasing functions, namely the inconvenience cost for rideshare drivers will increase when there are more rideshare passengers. This is because when there are more rideshare passengers, the rideshare drivers will need more detours for picking up or dropping off passengers and suffer longer waiting time, which leads to higher inconvenience cost for rideshare drivers. Note that this assumption is from an aggregate level, and it is consistent with existing literature such as Xu et al. (2015b), Ma et al. (2020), and Ma et al. (2022).

- **Inconvenience cost of rideshare passengers**: Similar to rideshare drivers, the rideshare service could also cause some inconvenience for rideshare passengers. The inconvenience cost may include the waiting time for drivers to pick them up, possible detour together with the drivers for picking up or dropping off other passengers, or even the anxiety to share a ride with strangers. The inconvenience cost of rideshare passengers is given by

\[
I_{a;rp}^{am}(f_{a}) \quad \forall a \in A_{rp}
\]

(3)

where \(I_{a;rp}^{am}(\bullet)\) and \(I_{a;rp}^{pm}(\bullet)\) are monotone increasing functions. When there are more rideshare passengers in the system, the rideshare passengers will need longer waiting time for rideshare drivers to pick them up and will possibly suffer more discomfort for sharing the ride, which leads to higher inconvenience cost for rideshare passengers.

- **Inconvenience cost of ride-hailing passengers**: Similar to rideshare passengers, ride-hailing passengers also experience some inconvenience, which may come from waiting for picking up after calling a ride. The inconvenience cost of ride-hailing passengers is as follows

\[
I_{a;hp}^{am}(f_{a}) \quad \forall a \in A_{hp}
\]

(4)

where \(I_{a;hp}^{am}(\bullet)\) and \(I_{a;hp}^{pm}(\bullet)\) are monotone increasing functions, namely the inconvenience cost of ride-hailing passengers increases as there are more ride-hailing passengers. The reason is that when there are more ride-hailing passengers in the system, there is more demand for this mode type, and as a result, the ride-hailing passengers experience longer waiting time, resulting in larger inconvenience cost.

- **Inconvenience cost of transit passengers**: Similar to rideshare and ride-hailing passengers, transit passengers suffer from inconvenience, which may be incurred by waiting for the public transit service, crowdedness in the transit vehicle, and even the walking distance between the transit stations and the origins (and/or the destinations). The inconvenience cost of transit passengers is defined as below

\[
I_{a;tp}^{am}(f_{a}) \quad \forall a \in A_{tp}
\]

(5)
where $I_{am}^{dp}(\bullet)$ and $I_{pm}^{dp}(\bullet)$ are monotone increasing functions, i.e., when there are more transit passengers, the inconvenience cost of transit passengers increases. This can be explained by that with more transit passengers, the waiting time for public transit service will be longer and it will be more crowded within the transit vehicle. Thus, transit passengers also experience larger inconvenience cost.

- **Payment of ride-hailing passengers:** The main reason that the ride-hailing drivers may be willing to provide ride-hailing service and may want to pick up more passengers is that they can receive compensation to cover part of their driving cost from each of the ride-hailing passengers. The payment for each ride-hailing passenger, which may be different in the morning and evening, is defined as

$$
\begin{align*}
R_{am}^{dp}(f_{am}) & \triangleq B_{am}^{dp} - S_{am}^{dp}(f_{am}) + E_{am}^{dp}(f_{am}) \\
R_{pm}^{dp}(f_{pm}) & \triangleq B_{pm}^{dp} - S_{pm}^{dp}(f_{pm}) + E_{pm}^{dp}(f_{pm})
\end{align*}
$$

where $B_{am}^{dp}$ and $B_{pm}^{dp}$ are positive constants; $S_{am}^{dp}(\bullet)$, $S_{pm}^{dp}(\bullet)$, $E_{am}^{dp}(\bullet)$ and $E_{pm}^{dp}(\bullet)$ are monotone increasing functions. The first part of Equation (6) is the benchmark price of rideshare service for arc $a \in A_{rp}$. The second and third part of Equation (6) are related to the relationship between supply and demand of the rideshare market: when there are more rideshare drivers, i.e., the supply of the rideshare market is larger, the price for rideshare service will decrease; when there are more rideshare passengers, namely the demand of the rideshare market becomes larger, the payment for rideshare service will increase.

- **Income of rideshare drivers:** Rideshare driver’s income is equal to the summation of all payments of rideshare passengers in his/her car. The actual number of passengers per vehicle for each arc is not predetermined, but should be within the range $[1, M]$. That is to say, on average there will be at least one passenger and at most $M$ passengers in each vehicle for each arc. Similar to Xu et al. (2015b), for simplicity we set the average number of rideshare passengers per rideshare vehicle for each arc to be fixed constants $\kappa_{am}, \kappa_{pm} \in [1, M]$, namely,

$$
\begin{align*}
R_{am}^{dp}(f_{am}) & \triangleq \kappa_{am} R_{am}^{dp}(f_{am}) \\
R_{pm}^{dp}(f_{pm}) & \triangleq \kappa_{pm} R_{pm}^{dp}(f_{pm})
\end{align*}
$$

- **Payment of ride-hailing passengers:** Similar to ride-hailing passengers, ride-hailing passengers also need to pay for using the ride-hailing service. The payment for each ride-hailing passenger is

$$
\begin{align*}
R_{am}^{hp}(f_{am}) & \triangleq B_{am}^{hp} + E_{am}^{hp}(f_{am}) \\
R_{pm}^{hp}(f_{pm}) & \triangleq B_{pm}^{hp} + E_{pm}^{hp}(f_{pm})
\end{align*}
$$

where $B_{am}^{hp}$ and $B_{pm}^{hp}$ are positive constants that represent the benchmark prices of ride-hailing service for arc $a \in A_{hp}$; $E_{am}^{hp}(\bullet)$ and $E_{pm}^{hp}(\bullet)$ are monotone increasing functions.

- **Payment of transit passengers:** Similar to ride-hailing and ride-hailing passengers, transit passengers will pay to use the public transit service. The payment for each transit passenger is given by

$$
\begin{align*}
R_{am}^{tp} \\
R_{pm}^{tp}
\end{align*}
$$
where $R_{a_{tp}}^{am}$ and $R_{a_{tp}}^{pm}$ are positive constants. Different from that of rideshare and ride-hailing passengers, the payment of transit passengers for arc $a \in A_{tp}$ is fixed and independent of the flows of transit passengers.

### 2.5 The overall arc and path cost functions

Denote $\psi$ as the conversion factor of time (minutes) to money (dollars); denote $r_{a_{am}}$ and $r_{a_{pm}}$ as the morning and evening travel time on arc $a \in A_{tp}$ experienced by transit passengers, respectively. Based on Sections 2.3 and 2.4, in sum, each traveler on arc $a \in A$ experiences a total cost of

$$
t_{ca}^{am}(f^{am}) =
\begin{cases}
\psi \times t_{a0}^{am}(f^{am}) & \forall a \in A_{sd} \text{ with } a_0 = T_0(a) \\
\psi \times t_{a0}^{am}(f^{am}) + I_{a_{rd}}^{am}(f^{am}) - R_{a_{rd}}^{am}(f^{am}) & \forall a \in A_{sd} \text{ with } a_0 = T_0(a) \\
\psi \times t_{a0}^{am}(f^{am}) + I_{a_{rp}}^{am}(f^{am}) + R_{a_{rp}}^{am}(f^{am}) & \forall a \in A_{tp} \text{ with } a_0 = T_0(a) \\
\psi \times t_{a0}^{am}(f^{am}) + I_{a_{hp}}^{am}(f^{am}) + R_{a_{hp}}^{am}(f^{am}) & \forall a \in A_{hp} \text{ with } a_0 = T_0(a) \\
\psi \times r_{a_{am}} + I_{a_{tp}}^{am}(f^{am}) + R_{a_{tp}}^{am} & \forall a \in A_{tp}
\end{cases}
$$

(10)

Each traveler needs to choose which path to travel on and which travel mode to use both in the morning and in the evening. In the extended coupled morning-evening network, it is equivalent to saying that each traveler will choose a morning-evening path $p \in P$ through the entire day. The coupled morning-evening path is illustrated in Fig. 3, which includes the path from home to the work place in the morning, the path from the work place to return home in the evening, and the chosen travel modes for both the morning and the evening. Denote $z_p$ as the flow of travelers of the coupled morning-evening path $p \in P$; denote $\delta_{a_{zp}}^{am}$ and $\delta_{a_{zp}}^{pm}$ as the arc-path indicators in the morning and in the evening, respectively. Let $z = \{z_p\}_{p \in P}$, $\Delta^{am} \triangleq \left(\delta_{a_{zp}}^{am}\right)_{(a,p) \in A \times P}$ and $\Delta^{pm} \triangleq \left(\delta_{a_{zp}}^{pm}\right)_{(a,p) \in A \times P}$.

Similar to the path cost structure of Xu et al. (2015b) and Li et al. (2020), the total cost experienced by a traveler throughout the entire day on path $p \in P$ can be represented as

$$
TC_p(z) = \sum_{a \in A} \left[\delta_{a_{zp}}^{am} \times t_{ca}^{am}(f^{am}) + \delta_{a_{zp}}^{pm} \times t_{ca}^{pm}(f^{pm})\right]
= \left(\Delta^{am}\right)^T t_{ca}^{am}(\Delta^{am} z) + \left(\Delta^{pm}\right)^T t_{ca}^{pm}(\Delta^{pm} z)
$$

(11)
2.6 Rideshare capacity constraints

Denote the passenger capacity of each rideshare vehicle as $M$. We have the rideshare capacity constraints as follows,

- in terms of traffic flows:

$$
\begin{align*}
    f_{a \text{rd}}^{\text{am}} &\leq f_a \leq M \times f_{a \text{rd}}^{\text{am}} \\
    f_{a \text{rd}}^{\text{pm}} &\leq f_a \leq M \times f_{a \text{rd}}^{\text{pm}}
\end{align*}
$$

$\forall a \in A_{\text{rp}}$ with $a_{\text{rd}} = T_{\text{rd}}(T_0(a))$ (12)

- or equivalently, in terms of passenger counts:

$$
\begin{align*}
    \sum_{p \in P^{\text{am}}, a_{\text{rd}} \in \mathcal{P}} z_p &\leq \sum_{p \in P^{\text{pm}}, a_{\text{rd}} \in \mathcal{P}} z_p \leq M \sum_{p \in P^{\text{am}}, a_{\text{rd}} \in \mathcal{P}} z_p \\
    \sum_{p \in P^{\text{pm}}, a_{\text{rd}} \in \mathcal{P}} z_p &\leq \sum_{p \in P^{\text{pm}}, a_{\text{rd}} \in \mathcal{P}} z_p \leq M \sum_{p \in P^{\text{pm}}, a_{\text{rd}} \in \mathcal{P}} z_p
\end{align*}
$$

$\forall a \in A_0$, $a_{\text{rd}} = T_{\text{rd}}(a)$, and $a_{\text{rp}} = T_{\text{rp}}(a)$ (13)

These constraints ensure two things on each arc: (i) the total number of rideshare drivers does not exceed the total number of rideshare passengers; (ii) the total number of rideshare passengers is no more than $M$ times the total number of rideshare drivers due to the vehicular capacity.

2.7 Demand satisfaction and flow conservation equations

The demand satisfaction equations are used to balance total trip demands with path flows and ensure morning trip demands equal evening trip demands; namely, at the aggregate, the total trip demands are derived from the passenger trips between OD pairs:

$$
D_k = \sum_{p \in P_k} z_p \quad \forall k \in \mathcal{K}
$$

(14)

The above path-based demand satisfaction equations [14] can be reformulated by arc flows, but extra flow conservation equations will be needed. First, we decompose the morning arc flows $f_a^{\text{am}}$ and evening arc flows $f_a^{\text{pm}}$ by morning OD pair $k \in \mathcal{K}$ and evening return OD pair $\bar{k}$, respectively.

Denote $x_{a \text{; } k}^{\text{am}} \geq 0$ and $x_{a \text{; } \bar{k}}^{\text{pm}} \geq 0$ as the amount of flow for OD pair $k \in \mathcal{K}$ on arc $a \in A$ in the morning.
and the amount of flow for associated return OD pair \( \bar{k} \) on arc \( a \in A \) in the evening, respectively.

Then we have the arc flow decomposition as follows:

\[
f_{am}^a = \sum_{k \in K} x_{am}^{a,k} \quad \text{and} \quad f_{pm}^a = \sum_{k \in K} x_{pm}^{a,k} \quad \forall a \in A
\]  

(15)

The demand satisfaction equations can then be represented from the perspective of arc flows as follows:

\[
\sum_{a \in \text{IN}(d_k)} x_{am}^{a,k} = \sum_{a \in \text{OUT}(o_k)} x_{am}^{a,k} = \sum_{a \in \text{IN}(d_{\bar{k}})} x_{pm}^{a,k} = \sum_{a \in \text{OUT}(o_{\bar{k}})} x_{pm}^{a,k} = D_k
\]

(16)

where \( o_k \) and \( d_k \) represent the origin and destination of OD pair \( k \) in the morning; \( o_{\bar{k}} \) and \( d_{\bar{k}} \) represent the origin and destination of the associated return OD pair \( \bar{k} \) in the evening; \( \text{IN}(i) \) and \( \text{OUT}(i) \) represent the sets of arcs entering and leaving node \( i \in N \), respectively.

We also need the flow conservation equations below for the nodes other than origins and destinations. The flow conservation equations guarantee the inflow of a node \( i \in N \) is equal to the outflow of that node, which can be formulated as follows:

\[
\begin{cases}
\sum_{a \in \text{IN}(i)} x_{am}^{a,k} - \sum_{a \in \text{OUT}(i)} x_{am}^{a,k} = 0 & \forall i \in N \setminus \{o_k, d_k\}, \forall k \in K \\
\sum_{a \in \text{IN}(i)} x_{pm}^{a,k} - \sum_{a \in \text{OUT}(i)} x_{pm}^{a,k} = 0 & \forall i \in N \setminus \{o_{\bar{k}}, d_{\bar{k}}\}, \text{associated } \bar{k}
\end{cases}
\]

(17)

Under the assumption that the morning drivers will remain to be drivers in the evening, we have the following expression equating the total numbers of morning and evening drivers:

\[
\sum_{a_{sd}, a_{rd} \in \text{OUT}(o_k)} (x_{am}^{a_{sd};k} + x_{am}^{a_{rd};k}) = \sum_{a_{sd}, a_{rd} \in \text{IN}(d_{\bar{k}})} (x_{pm}^{a_{sd};\bar{k}} + x_{pm}^{a_{rd};\bar{k}}) \quad \forall k \in K \text{ and associated } \bar{k}
\]

(18)

The demand satisfaction equations (16) together with the driver flow conservation equations (18) guarantee that the total number of morning and evening passengers are equal. This is consistent with the fact that if a traveler chooses to be a (rideshare or ride-hailing or transit) passenger in the morning, (s)he will have to take a ride or the transit back home in the evening.

2.8 The overall equilibrium model

In this section, we summarize the aforementioned sections and present a general equilibrium model to capture the complicated interactions between solo drivers, rideshare drivers, rideshare passengers, ride-hailing passengers, and transit passengers that allows travelers to switch between different transportation modes in a coupled morning-evening commute framework.

Based on path flows and path cost functions, the model is formulated as a variational inequality (VI) defined by the pair of mapping \( \Phi \) and feasible set \( \mathcal{HF} \), notated as \( \text{VI}(\Phi, \mathcal{HF}) \), as follows:
\[ \Phi(z) \triangleq \left( (T_{C_p}(z))_{p \in P} \right), \]

\[ \mathcal{H} \triangleq \left\{ z \triangleq (z_p)_{p \in P} \geq 0 \right\}, \]

\[ \mathcal{H}_F \triangleq \left\{ z \triangleq (z_p)_{p \in P} \geq 0 \right\}, \]

Similarly, based on arc flows and arc cost functions, the model is formulated as a VI defined by the pair of mapping \( \Phi' \) and feasible set \( \mathcal{H}_F' \), notated as \( \text{VI}(\Phi', \mathcal{H}_F') \), as follows:

\[ \Phi'(f) \triangleq \left( (tc_{a}^{am}(f_{a})^{am}), tc_{a}^{pm}(f_{a})^{pm}) \right)_{a \in A}, \]

\[ \mathcal{H}_F' \triangleq \left\{ f \triangleq \left( f_{a}^{am/pm} \right)_{a \in A} \left| \exists x_{a}^{am} \triangleq \left\{ x_{a,k}^{am,pm} \right\} \geq 0 \text{ and } \right. \right. \]

\[ \exists x_{a}^{pm} \triangleq \left\{ x_{a,k}^{pm} \right\} \geq 0 \text{ satisfying } (12), (15), (16), (17), (18) \].

Remarks:

(i) By Definition 1.1.1 of Facchinei and Pang (2003), variational inequality \( \text{VI}(\Phi, \mathcal{H}) \) is the problem to find a vector \( z \in \mathcal{H} \) such that \( (z - z')^T \Phi(z) \geq 0 \forall z' \in \mathcal{H} \).

(ii) The path-based \( \text{VI}(\Phi, \mathcal{H}) \) and the arc-based \( \text{VI}(\Phi', \mathcal{H}_F') \) are equivalent. The proof is as follows:

\[ \left( f - f' \right)^T \Phi'(f) \geq 0 \forall f' \in \mathcal{H}_F' \]

\[ \iff (z - z')^T \Phi'(z) \geq 0 \forall z' \in \mathcal{H} \]

\[ \iff (z - z')^T \Delta^T \Phi'(z) \geq 0 \forall z' \in \mathcal{H} \]

\[ \iff (z - z')^T \Phi(z) \geq 0 \forall z' \in \mathcal{H} \]

\[ \iff \Delta(z - z)^T \Phi(z) \geq 0 \forall z' \in \mathcal{H} \]

where \( \Delta \triangleq \left( \delta_{a,p} \right)_{(a,p) \in A \times P} \) and \( \delta_{a,p} \) are the arc-path indicators. Note that the relation between the arc flows and passenger counts is implicitly (but not explicitly) expressed by \( f = \Delta z \), which will appear in Remark (v) below. Also, the \( \Delta \) here is for the coupled morning-evening network in Fig. 3, which contains the information of both \( \Delta_{am} \) and \( \Delta_{pm} \);

(iii) Since the mapping \( \Phi \) is continuous and the set \( \mathcal{H} \) is compact and convex, by Corollary 2.2.5 of Facchinei and Pang (2003) it follows that the path-based \( \text{VI}(\Phi, \mathcal{H}) \) has a solution. Similarly, we can also show that the arc-based \( \text{VI}(\Phi', \mathcal{H}_F') \) has a solution. Thus so does our coupled morning-evening traffic equilibrium model;

(iv) We provide the conditions on the model parameters under which the arc-based \( \text{VI}(\Phi', \mathcal{H}_F') \) will have a unique solution, namely when the equilibrium of the proposed model will be unique. The details can be found in Proposition 1 in Appendix 1;

(v) Since the arc-based \( \text{VI}(\Phi', \mathcal{H}_F') \) has a unique solution, its equivalence to the path-based \( \text{VI}(\Phi, \mathcal{H}) \) is in the sense that they will derive the same unique arc flows \( f \). Note that for the path-based \( \text{VI}(\Phi, \mathcal{H}) \), the flow of an arc is calculated as the summation over the flows of all paths across this arc, i.e., \( f = \Delta z \). However, the path flows \( z \) obtained from the path-based \( \text{VI}(\Phi, \mathcal{H}) \) may not be unique. This is consistent with the traditional traffic equilibrium models (e.g., Sheffi, 1985): Although the path flows derived from the path-based model will not be unique, different path flow
solutions will lead to the same unique arc flows, which can also be derived from the equivalent arc-based model:

(vi) The path flows derived from the path-based model may not be unique. Thus, by solving the path-based model, it is possible to derive a solution in which a rideshare passenger is served by multiple rideshare drivers along his/her path, namely, there are transfers of rideshare passengers. However, given a feasible equilibrium solution of path flows with transfers of rideshare passengers, we can always construct a feasible equilibrium solution of path flows without transfers from it. Besides, these two path flow solutions correspond to the same arc flows at the aggregate level and thus the same equilibrium state, which can also be obtained by solving the equivalent arc-based model. For more details of the construction, please refer to Proposition 2 in Appendix 2. Moreover, the transfers of rideshare passengers will not happen in the numerical experiments since we are solving the arc-based model.

2.9 The extended user equilibrium conditions

Using the VI formulation, we present the mathematical properties in Section 2.8, such as existence and uniqueness of an equilibrium, and provide the proof in Appendix 1. In order to analyze the extended user equilibrium conditions in this section, we need to convert the VI formulation into the equivalent mixed complementarity problem (MiCP) formulation. The overall path-based MiCP formulation and arc-based MiCP formulation can be found in Appendix 3 and Appendix 4, respectively. The arc-based MiCP formulation is also used to solve the proposed model in Section 3.

We proposed an extended user equilibrium principle that describes a complementary relation between the daily commute path flows and the travelers’ minimum disutilities; it is based on the combined morning-evening round trips, allowing the switches of commute types. This type of equilibrium distinguishes itself from the separate morning or evening commute. The disutilities pertain to each OD pair \(k \in K\) and path flows \(z_p\) of all the morning-evening path \(p \in P_k\) in the extended network connecting that OD pair. That is to say, for each OD pair \(k\), the chosen morning-evening path \(p \in P_k\) connecting this OD pair among the 10 travel modes in Fig. 1 will all have travel costs equal to the minimum travel cost for all paths of the OD pair in question, and this common cost does not exceed the travel costs of the unchosen morning-evening paths for the travel mode joining the same OD pair. This extends Wardrop’s user equilibrium principle for the coupled morning-evening commute instead of the separate morning or evening commute in a traditional traffic equilibrium problem.

Another difference compared to the traditional user equilibrium is that travelers’ total cost includes not only the path-based cost defined in Section 2.5, but also the additional costs induced by the rideshare capacity constraints in Section 2.6. Denote \(\perp\) as the perpendicularity notation, \(x \perp y \iff x^T y = 0\) (see Definition 1.1.5 of Facchinei and Pang (2003)). Then from the equivalent path-based MiCP formulation in Appendix 3 or the equivalent arc-based MiCP formulation in Appendix 4, the equivalent complementarity formulation of the Inequalities (12) can be written as follows:

\[
\begin{align*}
0 \leq \eta^{am}_{a} & \perp f^{am}_{a} - f^{am}_{a} \geq 0 \\
0 \leq \eta^{am}_{a} & \perp M f^{am}_{a} - f^{am}_{a} \geq 0 \\
0 \leq \eta^{pm}_{a} & \perp f^{pm}_{a} - f^{pm}_{a} \geq 0 \\
0 \leq \eta^{pm}_{a} & \perp M f^{pm}_{a} - f^{pm}_{a} \geq 0
\end{align*}
\forall a \in A_0,
\]

\(a_{rd} = T_{rd}(a),\)

and \(a_{rp} = T_{rp}(a)\)

(19)
where the variables $\eta^\pm_{a}^{am}$ and $\eta^\pm_{a}^{pm}$ are the additional costs induced by the morning and evening rideshare capacity constraints, respectively. Based on the market clearance in economics, the rideshare capacity constraints describe only the actual flows of rideshare drivers and passengers, while the potential flows of rideshare drivers and passengers are suppressed by these additional costs. In practice, the additional costs $\eta^\pm_{a}^{am}$ and $\eta^\pm_{a}^{pm}$ can be considered as part of the compensations and surge prices to balance supply and demand in the rideshare market. For example, only when $f^am_{a_{rd}} = f^am_{a_{rp}}$ for some $a_{rd} = T_{rd}(a)$, $a_{rp} = T_{rp}(a)$ and $a \in A_0$, namely the flow on rideshare-driver arc equals to the relevant flow on rideshare-passenger arc, there could be a compensation $\eta^+_{a}^{am} > 0$ incurred to avoid the situation that $f^am_{a_{rd}} > f^am_{a_{rp}}$, namely there is more potential supply than demand; similarly, only when $M \times f^am_{a_{rd}} = f^am_{a_{rp}}$ for some $a_{rd} = T_{rd}(a)$, $a_{rp} = T_{rp}(a)$ and $a \in A_0$, namely the rideshare-passenger arc is at capacity, then there could be an extra payment $\eta^-_{a}^{am} > 0$ incurred to prevent the situation that $f^am_{a_{rp}} > M \times f^am_{a_{rd}}$, i.e., there is more potential demand than supply. Note that similar additional costs are also used in other traffic equilibrium models with rideshare services (e.g., Xu et al., 2015b; Ma et al., 2020; Li et al., 2020) and they are explained in a similar way.

Denote $u_k$ as the generalized (least) disutility of OD pair $k \in K$, which is the minimum generalized travel cost under an equilibrium state for all paths connecting OD pair $k \in K$. Written as the equivalent complementarity formulation of path-based cost function (11) based on the equivalent path-based MiCP formulation in Appendix 3, the extended path-based user equilibrium conditions for the coupled morning and evening commutes among the 10 types of travel modes in Fig. 1 are:

$$0 \leq z_p \perp TC_p(z) - \frac{\lambda_p(\eta_a)}{\text{compensations}} - \frac{u_k}{\text{least disutility}} \geq 0, \quad \forall p \in P_k, \forall k \in K \quad (20)$$

where, in this context, the perpendicularity notation $\perp$ asserts the complementarity between the coupled morning-evening path flows and the travelers’ deviations from the least disutilities. In other words, if a traveler chooses the morning-evening path $p \in P_k$, then the path cost/disutility must be the minimum of all costs for all paths connecting this OD pair $k \in K$. Denote $p^{am} \in P_k^{am}$ as the morning path of $p$, $p^{pm} \in P_k^{pm}$ as the evening path of $p$. Let $\eta_a \triangleq (\eta^+_{a}^{am}, \eta^-_{a}^{am}, \eta^+_{a}^{pm}, \eta^-_{a}^{pm})_{a \in A_0}$, here for all $p \in P_k = P_k^{am} \times P_k^{pm}$ we have that,

$$\lambda_p(\eta_a) \triangleq \sum_{T_{rd}(a) \in P_{am} \cap A_{rd}} \left(M \eta^-_{a}^{am} - \eta^+_{a}^{am}\right) + \sum_{T_{rd}(a) \in P_{pm} \cap A_{rd}} \left(M \eta^-_{a}^{pm} - \eta^+_{a}^{pm}\right)$$

$$+ \sum_{T_{rp}(a) \in P_{am} \cap A_{rp}} \left(\eta^+_{a}^{am} - \eta^-_{a}^{am}\right) + \sum_{T_{rp}(a) \in P_{pm} \cap A_{rp}} \left(\eta^+_{a}^{pm} - \eta^-_{a}^{pm}\right)$$

The extended user equilibrium conditions can also be formulated from an arc perspective. That is to say, for each OD pair $k$, the chosen arc $a \in A$ in the extended network connecting this OD pair among the 10 travel modes in Fig. 1 will all have travel costs equal to the least disutility of the OD pair in question, and this common cost does not exceed the travel costs of the unchosen arcs for the travel mode joining the same OD pair. One difference compared to the extended path-based user equilibrium is that travelers’ total cost includes not only the arc-based costs defined in Section 2.5, the compensations $(\eta^\pm_{a}^{am}, \eta^\pm_{a}^{pm})$ induced by the rideshare capacity constraints in Section 2.6, but also the multipliers $(\mu^k_{\pm}^{am}, \mu^k_{\pm}^{pm}, \zeta_{a;k})$ induced by demand satisfaction and flow conservation equations in Section 2.7. Based on the equivalent arc-based MiCP formulation in Appendix 4, the
extended arc-based user equilibrium conditions for the coupled morning and evening commutes are:

\[
0 \leq x_{ai,k}^{am} \pm t\alpha_{ai,k}^{am}(x_{ai,k}^{am}) + \omega_{ai,k}^{am} \eta_{T(a)}^{am} + \omega_{ai,k}^{-am} \eta_{T(a)}^{-am} - \mu_{i}^{k;am} - \mu_{j}^{k;am} - \omega_{ai,k}^{am} s_{ai,k} \geq 0,
\]

\[
\forall a = (i, j) \in A, \forall k \in K
\]

\[
0 \leq x_{ai,k}^{pm} \pm t\alpha_{ai,k}^{pm}(x_{ai,k}^{pm}) + \omega_{ai,k}^{pm} \eta_{T(a)}^{pm} + \omega_{ai,k}^{-pm} \eta_{T(a)}^{-pm} - \mu_{i}^{k;pm} - \mu_{j}^{k;pm} - \omega_{ai,k}^{pm} \zeta_{ai,k} \geq 0,
\]

\[
\forall a = (i, j) \in A, \text{ associated } \bar{k}
\]

where

\[
\omega_{ai,k}^{+am} = \omega_{ai,k}^{+pm} \triangleq \begin{cases} 
0, & \text{if } a \in A_{sd} \cup A_{hp} \cup A_{tp} \\
1, & \text{if } a \in A_{rd} \\
-1, & \text{if } a \in A_{tp}
\end{cases}
\]

\[
\omega_{ai,k}^{-am} = \omega_{ai,k}^{-pm} \triangleq \begin{cases} 
0, & \text{if } a \in A_{sd} \cup A_{hp} \cup A_{tp} \\
-M, & \text{if } a \in A_{rd} \\
1, & \text{if } a \in A_{tp}
\end{cases}
\]

\[
\omega_{ai,k}^{am} = \omega_{ai,k}^{pm} \triangleq \begin{cases} 
1, & \text{if } a \in A_{sd} \cup A_{rd} \\
0, & \text{if } a \in A_{tp} \cup A_{hp} \cup A_{tp}
\end{cases}
\]

Although the extended arc-based user equilibrium conditions (21) do not appear intuitive, it is equivalent to the extended path-based user equilibrium conditions (20). This comes from the equivalence of the path-based VI($\Phi, \mathcal{H}_{F}$) and the arc-based VI($\Phi', \mathcal{H}_{F}'$), which has been shown in Remark (ii) of Section 2.8. Since the feasible sets $\mathcal{H}_{F}'$ and $\mathcal{H}_{F}$ are polyhedra, their equivalent mixed complementarity problem (MiCP) formulations derived from Proposition 1.2.1 of Facchinei and Pang (2003) must also be equivalent. For the overall equivalent MiCP of the arc-based VI model, we refer to Appendix 4, which provides more details about how the multipliers ($\mu_{i}^{k;am}$, $\mu_{i}^{k;pm}$, $\zeta_{ai,k}$) are induced.

3 Computational Results

In this section, we use the well-known Sioux-Falls network to test the proposed model. The numerical experiments are derived by solving the equivalent arc-based MiCP formulation of the arc-based VI model in Section 2.8 (see Appendix 4). The results in this section are obtained by solving the MiCP using Knitro (Byrd et al. 2006) on the NEOS server.

For the experiments, we use functions for inconvenience costs and payments similar to those defined by Xu et al. (2015b) as follows:

- **Inconvenience cost of rideshare drivers:**

\[
\begin{align*}
I_{ai,k}^{am}(f_{ai,k}^{am}) & \triangleq \gamma_{ai,k}^{am} f_{ai,k}^{am} \\
I_{ai,k}^{pm}(f_{ai,k}^{pm}) & \triangleq \gamma_{ai,k}^{pm} f_{ai,k}^{pm}
\end{align*}
\]

\[
\forall a \in A_{rd}
\]
where $a_{rp} = T_{rp}(T_0(a))$ is the rideshare-passenger arc corresponding to the rideshare-driver arc $a \in A_{rd}$. The constants $\gamma_{rd}^{am}$ and $\gamma_{rd}^{pm}$ are positive.

- **Inconvenience cost of rideshare passengers**:

$$ I_{a \rightarrow rp}(f^{am}) \triangleq \gamma_{rp}^{am} f_a \quad \forall a \in A_{rp} $$

$$ I_{a \rightarrow rp}(f^{pm}) \triangleq \gamma_{rp}^{pm} f_a \quad \forall a \in A_{rp} $$

(23)

where the constants $\gamma_{rp}^{am}$ and $\gamma_{rp}^{pm}$ are positive.

- **Inconvenience cost of ride-hailing passengers**:

$$ I_{a \rightarrow hp}(f^{am}) \triangleq \gamma_{hp}^{am} f_a \quad \forall a \in A_{hp} $$

$$ I_{a \rightarrow hp}(f^{pm}) \triangleq \gamma_{hp}^{pm} f_a \quad \forall a \in A_{hp} $$

(24)

where the constants $\gamma_{hp}^{am}$ and $\gamma_{hp}^{pm}$ are positive.

- **Inconvenience cost of transit passengers**:

$$ I_{a \rightarrow tp}(f^{am}) \triangleq \gamma_{tp}^{am} f_a \quad \forall a \in A_{tp} $$

$$ I_{a \rightarrow tp}(f^{pm}) \triangleq \gamma_{tp}^{pm} f_a \quad \forall a \in A_{tp} $$

(25)

where the constants $\gamma_{tp}^{am}$ and $\gamma_{tp}^{pm}$ are positive.

- **Payment of rideshare passengers**:

$$ R_{a \rightarrow rp}(f^{am}) \triangleq \kappa_{am} R_{a \rightarrow rp}(f^{am}) $$

$$ R_{a \rightarrow rp}(f^{pm}) \triangleq \kappa_{pm} R_{a \rightarrow rp}(f^{pm}) $$

(26)

where $a_{rd} = T_{rd}(T_0(a))$ is the rideshare-driver arc corresponding to the rideshare-passenger arc $a \in A_{rp}$, $a_0 = T_0(a)$ is the original arc corresponding to the rideshare-driver arc $a \in A_{rp}$. The constants $\rho_{rp}^{am}$, $\rho_{rp}^{pm}$, $\nu_{rp}^{am}$, $\nu_{rp}^{pm}$, $\nu_{rp}^{am}$, and $\nu_{rp}^{pm}$ are positive. The parameters $t_{a_0}$ are the free flow travel time of arc $a_0 \in A_0$.

- **Income of rideshare drivers**:

$$ R_{a \rightarrow rd}(f^{am}) \triangleq \kappa_{am} R_{a \rightarrow rd}(f^{am}) $$

$$ R_{a \rightarrow rd}(f^{pm}) \triangleq \kappa_{pm} R_{a \rightarrow rd}(f^{pm}) $$

(27)

where $a_{rp} = T_{rp}(T_0(a))$ is the rideshare-passenger arc corresponding to the rideshare-driver arc $a \in A_{rd}$. The constants $\kappa_{am}$, $\kappa_{pm} \in [1, M]$.

- **Payment of ride-hailing passengers**:

$$ R_{a \rightarrow hp}(f^{am}) \triangleq \rho_{hp}^{am} t_{a_0} + w_{hp}^{am} f_a $$

$$ R_{a \rightarrow hp}(f^{pm}) \triangleq \rho_{hp}^{pm} t_{a_0} + w_{hp}^{pm} f_a $$

(28)
where \( a_0 = \mathcal{T}_0(a) \) is the original arc corresponding to the ride-hailing-passenger arc \( a \in \mathcal{A}_{hp} \). The constants \( \rho_{hp}^{\text{am}}, \rho_{hp}^{\text{pm}}, w_{hp}^{\text{am}}, \text{ and } w_{hp}^{\text{pm}} \) are positive. The parameters \( t_{a_0} \) are the free flow travel time of arc \( a_0 \in \mathcal{A}_0 \).

- **Payment of transit passengers:**

\[
\begin{align*}
R_{am}^{\text{tp}} & \triangleq \rho_{am}^{\text{tp}} r_{am}^{\text{tp}} \\
R_{pm}^{\text{tp}} & \triangleq \rho_{pm}^{\text{tp}} r_{pm}^{\text{tp}}
\end{align*}
\]  

\( \forall a \in \mathcal{A}_{tp} \) (29)

where \( \rho_{am}^{\text{tp}} \text{ and } \rho_{pm}^{\text{tp}} \) are positive. The parameters \( r_{am}^{\text{tp}} \text{ and } r_{pm}^{\text{tp}} \) are the travel time experienced by transit passengers on arc \( a \in \mathcal{A}_{tp} \) in the morning and evening, respectively.

Model parameters in terms of travel modes are set based on the following guidelines: (1) parameters for the inconvenience of rideshare passengers are no smaller than those of ride-hailing passengers, and parameters for the inconvenience of transit passengers are no smaller than those of rideshare passengers, e.g., \( \gamma_{am}^{\text{rd}} \geq \gamma_{am}^{\text{rp}} \geq \gamma_{am}^{\text{hp}} \); (2) parameters for the payment of rideshare passengers are no larger than those of ride-hailing passengers, and parameters for the payment of transit passengers are no larger than those of rideshare passengers, e.g., \( w_{am}^{\text{am}} \leq w_{am}^{\text{pm}} \leq w_{am}^{\text{hp}} \). In addition, we set the parameters to satisfy the conditions of Proposition 1 in Appendix 1 in order to guarantee solution uniqueness.

For the settings of the Sioux-Falls network, we follow Stabler (2020), including the geometry, travel demand for each OD pair, and parameters of the BPR function for each arc. The original network has 76 arcs and 24 nodes. After constructing the extended network, there will be 76 \( \times \) 5 = 380 arcs and 24 \( \times \) 4 = 96 nodes. There are 24 origins and 24 destinations in total in the network, which are all included in the computational experiments. The transit network is consistent with existing literature such as LeBlanc (1988), Gentile et al. (2005), and Larrain et al. (2021). Model parameters for the base case are listed in Table 4, which is from Xu et al. (2015b). The conversion factor of time to money, \( \psi \), is set to be 30 dollars/hour. The computation time for solving the problems in this section ranges from 0.82 to 5.66 hours.

### Table 4. Parameters of the base case.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma_{am}^{\text{rd}}, \gamma_{pm}^{\text{rd}} )</td>
<td>0.01</td>
<td>Dollars/Unit Flow</td>
</tr>
<tr>
<td>( \gamma_{am}^{\text{rp}}, \gamma_{pm}^{\text{rp}} )</td>
<td>0.01</td>
<td>Dollars/Unit Flow</td>
</tr>
<tr>
<td>( \gamma_{am}^{\text{hp}}, \gamma_{pm}^{\text{hp}} )</td>
<td>0.001</td>
<td>Dollars/Unit Flow</td>
</tr>
<tr>
<td>( \gamma_{am}^{\text{tp}}, \gamma_{pm}^{\text{tp}} )</td>
<td>0.1</td>
<td>Dollars/Unit Flow</td>
</tr>
<tr>
<td>( \kappa_{am}, \kappa_{pm} )</td>
<td>2</td>
<td>Persons</td>
</tr>
<tr>
<td>( \rho_{am}^{\text{am}}, \rho_{pm}^{\text{pm}} )</td>
<td>0.5</td>
<td>Dollars/Unit Time</td>
</tr>
<tr>
<td>( \psi_{am}^{\text{pm}}, \psi_{pm}^{\text{pm}} )</td>
<td>0.2</td>
<td>Dollars/Unit Flow</td>
</tr>
<tr>
<td>( w_{am}^{\text{am}}, w_{pm}^{\text{pm}} )</td>
<td>0.1</td>
<td>Dollars/Unit Flow</td>
</tr>
<tr>
<td>( \rho_{hp}^{\text{am}}, \rho_{pm}^{\text{pm}} )</td>
<td>0.5</td>
<td>Dollars/Unit Time</td>
</tr>
<tr>
<td>( w_{hp}^{\text{am}}, w_{hp}^{\text{pm}} )</td>
<td>0.15</td>
<td>Dollars/Unit Flow</td>
</tr>
<tr>
<td>( \rho_{am}^{\text{tp}}, \rho_{pm}^{\text{tp}} )</td>
<td>0.1</td>
<td>Dollars/Unit Time</td>
</tr>
</tbody>
</table>

Table 5 is derived from the base case, in which the morning and evening parameters are the same. It shows the travelers’ mode choice and mode switches in the morning commute and evening commute for the base case. We note that although the parameters for the morning and evening are the same, the travelers’ mode choice could be different. The reasons are that (i) the bi-directional roads are
not symmetrical for the morning and evening trips; (ii) the Sioux-Falls network is not symmetrical at the network level. The proposed model shows that 19.0% of the solo drivers in the morning switch to rideshare drivers in the evening, which leads to a cheaper rideshare price in the evening. As a result, 7.2% of ride-hailing or transit passengers in the morning switch to rideshare passengers in the evening.

Table 5. Travelers’ mode choice and mode switches.

<table>
<thead>
<tr>
<th></th>
<th>AM</th>
<th>PM</th>
</tr>
</thead>
<tbody>
<tr>
<td># Solo Drivers</td>
<td>95440</td>
<td>84413</td>
</tr>
<tr>
<td># Rideshare Drivers</td>
<td>58084</td>
<td>69111</td>
</tr>
<tr>
<td># Rideshare Passengers</td>
<td>130912</td>
<td>140330</td>
</tr>
<tr>
<td># Ride-hailing Passengers</td>
<td>62242</td>
<td>60910</td>
</tr>
<tr>
<td># Transit Passengers</td>
<td>14162</td>
<td>6077</td>
</tr>
<tr>
<td># AM Solo Drivers ⇒ PM Rideshare Drivers</td>
<td>11027 (19.0%)</td>
<td></td>
</tr>
<tr>
<td># AM Ride-hailing or Transit Passengers ⇒ PM Rideshare Passengers</td>
<td>9418 (7.2%)</td>
<td></td>
</tr>
</tbody>
</table>

3.1 Sensitivity analysis

Fig. 4 shows how the average Vehicle Miles Traveled (VMT) change when we change the parameter related to evening rideshare payment ($w_{rp}^{pm}$) and the parameter related to evening rideshare driver’s inconvenience ($\gamma_{rd}^{pm}$), respectively. As we can see, the change of evening parameters influences not only the evening average VMT but also the morning average VMT because of the interactions between the morning and evening commutes. When the evening parameters change, the evening traffic flows change; since the evening traffic flows have impact on the morning, the morning traffic flows also change; and from an aggregate level, we observe that the morning average VMT changes. For example, in Fig. 4(b) we can see that even if we only change the evening rideshare driver’s inconvenience, the morning average VMT changes rapidly. Without the coupled model, we can calculate the traffic equilibria for the morning and evening separately, which may overestimate the morning average VMT when we decrease the parameter related to evening rideshare payment ($w_{rp}^{pm}$) or the parameter related to evening rideshare driver’s inconvenience ($\gamma_{rd}^{pm}$).

Figure 4. Results of average Vehicle Miles Traveled when changing (a) $w_{rp}^{pm}$; (b) $\gamma_{rd}^{pm}$.
The coupling effects between morning and evening can also be observed in travelers’ mode choice. The changes of average VMTs in Fig. 4 can be explained by the changes of travelers’ mode choice, as shown in Fig. 5 and Fig. 6, respectively. Fig. 5 shows the sensitivity analysis for travelers’ mode choice when changing the rideshare payment \( w_{pm} \) in the evening. We can observe from Fig. 5(b) that as the evening rideshare payment increases, the number of rideshare passengers decreases. As a result, the rideshare market in the evening becomes smaller and we do not need so many rideshare drivers. More of the rideshare drivers and rideshare passengers switch to solo drivers than ride-hailing and transit passengers in the evening, which leads to an increase in the total number of drivers in the evening. With the coupling effects between morning and evening, the number of morning rideshare passengers decreases and the total number of morning drivers increases, as shown in Fig. 5(a). Here more travelers choose to drive in the morning because they know that it would be expensive to be a passenger in the evening. Since there are fewer rideshare drivers and passengers in the system, we can expect larger average VMTs as in Fig. 4(a).

Fig. 6 illustrates how the change of evening rideshare driver’s inconvenience \( \gamma_{rd} \) influences travelers’ mode choice both in the morning and evening. As shown in Fig. 6(b), when the evening rideshare driver’s inconvenience increases (perhaps because some drivers need to pick up their kids after work), the number of evening rideshare drivers decreases, which could cause higher rideshare payment in the evening. Consequently, there are fewer rideshare passengers in the evening. The rideshare drivers and passengers switch to solo drivers, ride-hailing passengers, and transit passengers in the evening. A similar phenomenon can be observed in Fig. 6(a) in the morning, due to the interactions between the morning and evening commutes. With fewer rideshare drivers and passengers in the system, we can expect larger average VMTs as in Fig. 4(b). Even if we only change the evening rideshare driver’s inconvenience, travelers’ mode choice in the morning changes rapidly. This is consistent with the increase of morning average VMT in Fig. 4(b), which again indicates that the influence of morning (or evening) parameters on evening (or morning) model results could be significant.
3.2 Comparison with a decoupled modeling approach

In Section 2, we have developed a model for coupled morning-evening commute, which we call the coupled model. In this section, we compare the equilibrium solution from the coupled model with a decoupled modeling approach.

Let the decoupled morning model be one that solves both the route and mode choice for the morning given a set of OD demands and the decoupled evening model be one that solves both the route and mode choice for the evening given a set of OD demands. Identifying an equilibrium solution to these two problems separately and then combining them together would most likely violate the constraint that if a traveler chooses to be a driver, (s)he must be a driver both in the morning and the evening. Thus, the two decoupled models must be linked in some manner.

Before presenting our approach for linking the models, we present two other models. Let the constrained decoupled morning model be one that solves the route choice and partial mode choice (e.g., rideshare, ride-hailing, and transit passengers) for the morning given a set of demands and a set of drivers for each OD pair and the constrained decoupled evening model be one that solves the route choice and partial mode choice (e.g., rideshare, ride-hailing, and transit passengers) for the evening given a set of demands and a set of drivers for each OD pair.

For a decoupled approach, we assume a traveler uses the decoupled morning model or the decoupled evening model to determine whether they become a driver or not. Based on this assumption, we present two decoupled solution approaches.

Solution D1

Step 1: Identify an equilibrium solution to the decoupled morning model to determine for each OD pair the set of routes and mode choice (solo drivers, rideshare drivers, rideshare passengers, ride-hailing passengers, and transit passengers) for the morning only. Mathematically, the decoupled morning model includes the morning part of Equations (12), (15), (16), and (17) as the feasible set, and the morning part of the function (10) as the mapping.
Step 2: Given the total number of drivers for each OD pair from the solution in Step 1, identify an equilibrium solution to the constrained evening model to determine for each OD the set of routes and mode choice (in particular, rideshare passengers, ride-hailing passengers, and transit passengers) for the evening only. Mathematically, the constrained evening model includes the evening part of Equations (12), (15), (16), (17), and the driver flow conservation constraint (18) as the feasible set, and the evening part of the function (10) as the mapping.

Solution D2

Step 1: Identify an equilibrium solution to the decoupled evening model to determine for each OD pair the set of routes and mode choice (solo drivers, rideshare drivers, rideshare passengers, ride-hailing passengers, and transit passengers) for the evening only. Mathematically, the decoupled evening model includes the evening part of Equations (12), (15), (16), and (17) as the feasible set, and the evening part of the function (10) as the mapping.

Step 2: Given the total number of drivers for each OD pair from the solution in Step 1, identify an equilibrium solution to the constrained morning model to determine for each OD the set of routes and mode choice (in particular, rideshare passengers, ride-hailing passengers, and transit passengers) for the morning only. Mathematically, the constrained morning model includes the morning part of Equations (12), (15), (16), (17), and the driver flow conservation constraint (18) as the feasible set, and the morning part of the function (10) as the mapping.

We next compare the equilibrium solution from the coupled model against the decoupled approach. The decoupled solution is based on solution D2, and similar results can be found for solution D1. The main quantities for comparison include least disutility, average VMT, and total number of drivers. Here the least disutility is the network-wide average, which is the weighted average in terms of OD demand for all the OD pairs.

We use the same parameters as those in Table 4, except that $\gamma_{pm}^{rd}$ is 0.036 dollars per unit flow. Recall $\gamma_{am}^{rd}$ is 0.010 dollars per unit flow. This scenario represents the case where the inconvenience cost of a rideshare driver from work place to home during the evening commute is higher than that from home to work place in the morning commute. Recall that in the decoupled solution D2, an individual will determine whether or not to be a driver using the decoupled evening model. With the total number of drivers fixed for each OD pair, then we can derive the morning equilibrium by solving the constrained morning model.

The comparison between the two models is shown in Table 6. We can see that the coupled model outputs a solution with a 10.0% smaller least disutility compared with the decoupled model. The decoupled model overestimates the number of drivers by 11.2% and the average VMT by 8.2% compared with the coupled model because the coupled model is capable of capturing the mode switches and interactions between morning and evening, which leads to fewer drivers and less average VMT in the system. As shown in Table 6, in this case, 4.4% of the morning rideshare passengers switch to ride-hailing or public transit services in the evening because of the higher cost of rideshare during the evening commute, due to, for example, some individuals need to pick up their children at their after-school activities, making the use of rideshare service during the evening less convenient. The decoupled model cannot capture this effect and most likely will predict that the traveler will drive to work, thus causing the overestimation of the number of drivers in this situation.

The advantage of the proposed coupled model over the decoupled model can also be observed when we change the evening rideshare driver’s inconvenience ($\gamma_{am}^{rd}$). The results of least utility and average VMT for both the coupled and decoupled models can be found in Fig. 7. For example, when the
evening rideshare driver’s inconvenience ($\gamma_{pm}^{rd}$) increases from 0.010 dollars to 0.036 dollars, the decoupled model overestimates the average VMT in the network by 5.8% - 8.1% compared with the coupled model.

Table 6. Comparison between coupled model and decoupled model.

<table>
<thead>
<tr>
<th></th>
<th>Coupled Model</th>
<th>Decoupled Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Least Disutility (Dollars)</td>
<td>85.58</td>
<td>95.04</td>
</tr>
<tr>
<td>Average VMT</td>
<td>13.00</td>
<td>14.04</td>
</tr>
<tr>
<td># Drivers</td>
<td>173749</td>
<td>193209</td>
</tr>
<tr>
<td># AM Rideshare Passengers</td>
<td>106766</td>
<td>99955</td>
</tr>
<tr>
<td># PM Rideshare Passengers</td>
<td>102065</td>
<td>94281</td>
</tr>
<tr>
<td># AM Ride-hailing Passengers</td>
<td>63651</td>
<td>57424</td>
</tr>
<tr>
<td># PM Ride-hailing Passengers</td>
<td>69576</td>
<td>63738</td>
</tr>
<tr>
<td># AM Transit Passengers</td>
<td>16675</td>
<td>10253</td>
</tr>
<tr>
<td># PM Transit Passengers</td>
<td>15451</td>
<td>9613</td>
</tr>
</tbody>
</table>

Figure 7. Comparison between coupled and decoupled models when changing $\gamma_{pm}^{rd}$: (a) Least Disutility; (b) Average Vehicle Miles Traveled.

4 Conclusions and Future Research

In this study, we include solo driving, rideshare, ride-hailing, and public transit as travel modes and integrate morning and evening commute trips in a general network equilibrium modeling framework, which allows travelers to switch from one type of commute mode in the morning to another in the evening, and allows passengers from different OD pairs to share a ride together. The model is formulated as a variational inequality (VI), and reformulated as an equivalent mixed complementarity problem (MiCP). Then the proposed model is evaluated on the Sioux-Falls network. The results show that the proposed coupled morning-evening traffic equilibrium model is capable of capturing the mode switches, and the coupling effects between morning and evening commutes. Our numerical examples show that considering morning and evening commutes separately tends to overestimate
the travelers’ disutility, number of drivers and average VMT in the network. For example, the proposed model produces 10.1% fewer drivers and 7.4% less average VMT in the system compared with a decoupled method when the rideshare price is higher in the evening commute than that of the morning commute. This is due to the coupling interaction effects between morning and evening commutes, e.g., rideshare passengers in the morning commute may switch to ride-hailing or transit passengers in the evening commute. When treating the morning and evening commutes separately, we cannot capture these interactions.

One direction for future research could be to extend our model to formulate the trip chains of travelers throughout an entire day. In our model, we already capture some trip chains of travelers, such as the home-work-home trip chain and the detours of rideshare vehicles. However, the whole-day trip chains of travelers have not been modeled explicitly. One challenge to model the entire-day trip chains is due to the probably much more complicated formulation. For example, after returning home by driving a car, a traveler may decide to take a ride for shopping at night, which will change the total number of drivers in the system. This may eventually lead to a non-square complementarity formulation, which is mathematically difficult to analyze. Another difficulty lies in how to solve the model. With the rideshare, ride-hailing, and public transit services providing more choices for travelers, the dimension of the problem will increase exponentially with longer trip chains. As a result, it could be quite challenging to solve the model. More resource is required for developing advanced scalable algorithms. Another research direction could be to extend our static traffic equilibrium model to a dynamic one. In this case, more advanced mathematical tools such as differential variational inequalities may be needed, which could lead to a rather different model.

Acknowledgements

We acknowledge METRANS for their financial support of this research.

References


Appendix 1. Uniqueness of the equilibrium.

In this section, we derive the conditions under which our proposed model will have a unique solution. We provide the condition on the model parameters under which the equilibrium will be globally unique. Furthermore, we show that, under the same condition, the equilibrium will be locally unique even when a commonly used assumption in the literature is violated.

In Proposition 1 below, we provide the conditions under which the arc cost functions \([10]\) will be strictly monotone, and as a result, our proposed model will have a unique solution. When all arcs in the network are used by the travelers, namely the situation that \(\theta^a \neq 0\) and \(\theta^m_a \neq 0\) for all \(a \in A_0\) in Proposition 1(i) below (note that this is a common assumption in existing literature such as Section 3.3 of Sheffi, 1985; Xu et al., 2015b; Ma et al., 2020; Li et al., 2020; Wang et al., 2021), we provide the conditions under which the model will have a globally unique solution. For the situation that some arcs are not used by travelers, namely when \(\theta^a = 0\) or \(\theta^m_a = 0\) for some \(a \in A_0\) as in Proposition 1(ii), global uniqueness will no longer be possible. Instead, we show that under the same condition, the local unique solution can be achieved.

**Proposition 1.** Under the following conditions:

\[
\begin{align*}
4t_a b \times \left( \frac{\sum_{k \in K} D_k}{\alpha_a} \right)^3 &< 4H^1_{am} H^m_{am} - 4(2^m_{am})^2 \quad \forall a \in A_0 \\
4t_a b \times \left( \frac{\sum_{k \in K} D_k}{\alpha_a} \right)^3 &< 4H^1_{pm} H^m_{pm} - 4(2^m_{pm})^2 \quad \forall a \in A_0
\end{align*}
\]

where \(H^1_{am} = \kappa_{am} \partial_{\alpha_a} s_{am} \rightleftharpoons \), \(H^m_{pm} = -\partial_{\alpha_a} s_{am} \rightleftharpoons \partial_{\alpha_a} I_{am} \rightleftharpoons \), \(H^m_{pm} = \partial_{\alpha_a} I_{am} \rightleftharpoons \partial_{\alpha_a} E_{am} \rightleftharpoons \) (here for simplicity, we denote \(\frac{\partial}{\partial f_{am} \rightleftharpoons} \) as \(\partial_t\)).

(i) If \(\theta^m_a \leq 4t_a b \times \left( \frac{f_{am} \rightleftharpoons}{\alpha_a} \right)^3 \neq 0\) and \(\theta^m_a \leq 4t_a b \times \left( \frac{f_{am} \rightleftharpoons}{\alpha_a} \right)^3 \neq 0\) for all \(a \in A_0\), with \(\alpha_{sd} = T_{sd}(a),\ \alpha_{rd} = T_{rd}(a),\ \alpha_{hp} = T_{hp}(a)\), we will have globally unique arc flows, namely \(f_{am} \rightleftharpoons, f_{am} \rightleftharpoons, f_{am} \rightleftharpoons, f_{am} \rightleftharpoons, f_{am} \rightleftharpoons, f_{am} \rightleftharpoons, f_{am} \rightleftharpoons,\) and \(f_{am} \rightleftharpoons\) are globally unique.

(ii) If \(\theta^m_a \leq 4t_a b \times \left( \frac{f_{am} \rightleftharpoons}{\alpha_a} \right)^3 = 0\) or \(\theta^m_a \leq 4t_a b \times \left( \frac{f_{am} \rightleftharpoons}{\alpha_a} \right)^3 = 0\) for some \(a \in A_0\) with \(\alpha_{sd} = T_{sd}(a),\ \alpha_{rd} = T_{rd}(a),\ \alpha_{hp} = T_{hp}(a)\), we will have locally unique arc flows, namely \(f_{am} \rightleftharpoons, f_{am} \rightleftharpoons, f_{am} \rightleftharpoons, f_{am} \rightleftharpoons, f_{am} \rightleftharpoons, f_{am} \rightleftharpoons, f_{am} \rightleftharpoons,\) and \(f_{am} \rightleftharpoons\) are locally unique.

**Proof.** We analyze the Jacobian matrix \(J \Phi(f^{am}, f^{pm})\) for the overall equilibrium model proposed in Section 2.8, which is a block diagonal with \(2 \times |A|\) diagonal blocks as follows,

\[
J \Phi'(f^{am}, f^{pm}) = \begin{bmatrix}
J \Phi_{\leftarrow am}(f^{am}) & 0 \\
0 & J \Phi_{\leftarrow pm}(f^{pm})
\end{bmatrix}
\]

where \(J \Phi_{\leftarrow am}(f^{am})\) is the Jacobian matrix for the morning arc flows, and \(J \Phi_{\leftarrow pm}(f^{pm})\) is the Jacobian matrix for the evening arc flows.

Note that the extended arc set is defined as \(\mathcal{A} \triangleq \mathcal{A}_{sd} \cup \mathcal{A}_{rd} \cup \mathcal{A}_{tp} \cup \mathcal{A}_{hp} \cup \mathcal{A}_{tp}\) in the extended network in Fig. 2, which includes a “solo-driver arc”, a “rideshare-driver arc”, a “rideshare-passenger arc”, a “ride-hailing-passenger arc”, and a “transit-passenger arc”. Thus, both \(J \Phi_{\leftarrow am}(f^{am})\) and
\( J \Phi f^{pm} \) consist of \( 5 \times |A_0| \) diagonal blocks \( B^{am} \). Take \( J \Phi f^{am} \) as an example, each block \( B^{am} \) is a Jacobian sub-matrix with respect to each arc \( a_0 \in A_0 \), which can be written as

\[
B^{am} = \begin{bmatrix}
\frac{\partial c^{am}_{0}}{\partial f^{am}_{a_0}} & \frac{\partial c^{am}_{1}}{\partial f^{am}_{a_0}} & \cdots & \frac{\partial c^{am}_{|A_0|}}{\partial f^{am}_{a_0}} \\
\frac{\partial c^{am}_{0}}{\partial f^{am}_{a_0}} & \frac{\partial c^{am}_{1}}{\partial f^{am}_{a_0}} & \cdots & \frac{\partial c^{am}_{|A_0|}}{\partial f^{am}_{a_0}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial c^{am}_{0}}{\partial f^{am}_{a_0}} & \frac{\partial c^{am}_{1}}{\partial f^{am}_{a_0}} & \cdots & \frac{\partial c^{am}_{|A_0|}}{\partial f^{am}_{a_0}}
\end{bmatrix}
\]

For simplicity, denote \( \frac{\partial}{\partial f^{am}_{a_0}} \) as \( \partial_a \), plug in Equation 10 then we have that

\[
B^{am} = \begin{bmatrix}
\vartheta^{am}_a & \vartheta^{am}_a & 0 & \vartheta^{am}_a & 0 \\
\vartheta^{am}_a & \vartheta^{am}_a + \kappa^{am} \partial_{td} S^{am}_{ar, rp} & \partial_{tp} f^{am}_{a, rd} - \kappa^{am} \partial_{tp} E^{am}_{a, rp} & \vartheta^{am}_a & 0 \\
\vartheta^{am}_a & \vartheta^{am}_a - \partial_{td} S^{am}_{ar, rp} & \partial_{tp} I^{am}_{a, rd} + \partial_{tp} E^{am}_{a, rp} & \vartheta^{am}_a & 0 \\
\vartheta^{am}_a & \vartheta^{am}_a & 0 & \vartheta^{am}_a + \partial_{hp} I^{am}_{a, hp} + \partial_{hp} E^{am}_{a, hp} & 0 \\
0 & 0 & 0 & 0 & \partial_{tp} f^{am}_{a, rp}
\end{bmatrix}
\]

where \( \vartheta^{am}_a = \frac{4a^{ab}}{a^{a}} \left( f^{am}_{ar, rd} + f^{am}_{ar, rd} + f^{am}_{ar, hp} \right)^3 \geq 0 \quad \forall a \in A_0 \) with \( a_{sd} = T_{sd}(a), a_{rd} = T_{rd}(a), a_{hp} = T_{hp}(a) \).

The matrix \( B^{am} \) is positive (semi)definite if its symmetric part of \( B^{am} = \frac{1}{2} \left( B^{am} + (B^{am})^T \right) \) is positive (semi)definite. \( \bar{B}^{am} \) can be calculated as follows:

\[
\bar{B}^{am} = \begin{bmatrix}
\vartheta^{am}_a & \vartheta^{am}_a & \vartheta^{am}_a & \vartheta^{am}_a & 0 \\
\vartheta^{am}_a & \vartheta^{am}_a + H_1^{am} & \vartheta^{am}_a & \vartheta^{am}_a & 0 \\
\vartheta^{am}_a & \vartheta^{am}_a + H_2^{am} & \vartheta^{am}_a & \vartheta^{am}_a & 0 \\
\vartheta^{am}_a & \vartheta^{am}_a & \vartheta^{am}_a & \vartheta^{am}_a & 0 \\
0 & 0 & 0 & 0 & H_5^{am}
\end{bmatrix}
\]

where \( H_1^{am} = \kappa^{am} \partial_{td} S^{am}_{ar, rp}, H_2^{am} = \frac{-\partial_{td} S^{am}_{ar, rp} + \partial_{tp} f^{am}_{a, rd} - \kappa^{am} \partial_{tp} E^{am}_{a, rp}}{2}, H_3^{am} = \partial_{tp} I^{am}_{a, rp} + \partial_{tp} E^{am}_{a, rp}, H_4^{am} = \partial_{hp} I^{am}_{a, hp} + \partial_{hp} E^{am}_{a, hp}, H_5^{am} = \partial_{hp} I^{am}_{a, hp} \).
Similarly, we can derive such matrix $\mathbf{\bar{B}}^{pm}$ for $J\Phi^{pm}(f^{pm})$ consisting of $\theta_a^{pm}$, $H_1^{pm}$, $H_2^{pm}$, $H_3^{pm}$, $H_4^{pm}$, $H_5^{pm}$. Below we provide the proof for matrix $\mathbf{\bar{B}}^{am}$, and the proof for matrix $\mathbf{\bar{B}}^{pm}$ is similar.

(i) Based on Theorem 2.3.3(a) and Proposition 2.3.2(b) of Facchinei and Pang (2003), we need the matrix $\mathbf{B}^{am}$ to be positive definite to derive globally unique arc flows, which is equivalent to show that all its upper left submatrices have positive determinants. The upper left $1 \times 1$ and $2 \times 2$ determinants of the matrix $\mathbf{B}^{am}$ are positive when $\theta^{am}_a > 0$. From that upper left $3 \times 3$ determinant of the matrix $\mathbf{B}^{am}$ is positive we have

\[
\begin{vmatrix}
\theta^{am}_a & \theta^{am}_a & \frac{\theta^{am}_a}{2} \\
\frac{\theta^{am}_a}{2} & \frac{\theta^{am}_a}{2} + H_1^{am} & \frac{\theta^{am}_a}{2} + H_2^{am} \\
\frac{\theta^{am}_a}{2} & \frac{\theta^{am}_a}{2} + H_2^{am} & \frac{\theta^{am}_a}{2} + H_3^{am}
\end{vmatrix} = -\frac{1}{4} \theta^{am}_a \left( H_1^{am} \theta^{am}_a - 4H_1^{am}H_3^{am} + 4(H_2^{am})^2 \right) > 0
\]

\[
\Rightarrow \theta^{am}_a < \frac{4H_1^{am}H_3^{am} - 4(H_2^{am})^2}{H_1^{am}}
\]

Under the condition above, the upper left $4 \times 4$ determinant of the matrix $\mathbf{B}^{am}$ is positive since

\[
\begin{vmatrix}
\theta^{am}_a & \theta^{am}_a & \frac{\theta^{am}_a}{2} & \frac{\theta^{am}_a}{2} + H_1^{am} \\
\frac{\theta^{am}_a}{2} & \frac{\theta^{am}_a}{2} + H_1^{am} & \frac{\theta^{am}_a}{2} + H_2^{am} & \frac{\theta^{am}_a}{2} + H_3^{am} \\
\frac{\theta^{am}_a}{2} & \frac{\theta^{am}_a}{2} + H_2^{am} & \frac{\theta^{am}_a}{2} + H_3^{am} & \frac{\theta^{am}_a}{2} + H_4^{am} \\
\frac{\theta^{am}_a}{2} & \frac{\theta^{am}_a}{2} + H_4^{am} & \frac{\theta^{am}_a}{2} + H_5^{am} & \frac{\theta^{am}_a}{2} + H_4^{am}
\end{vmatrix} = -\frac{1}{4} \times H_4^{am} \times \theta^{am}_a \left( H_1^{am} \theta^{am}_a - 4H_1^{am}H_3^{am} + 4(H_2^{am})^2 \right) > 0
\]

Similarly, the upper left $5 \times 5$ determinant of the matrix $\mathbf{B}^{am}$ is positive since

\[
\begin{vmatrix}
\theta^{am}_a & \theta^{am}_a & \frac{\theta^{am}_a}{2} & \frac{\theta^{am}_a}{2} + H_1^{am} & 0 \\
\frac{\theta^{am}_a}{2} & \frac{\theta^{am}_a}{2} + H_1^{am} & \frac{\theta^{am}_a}{2} + H_2^{am} & \frac{\theta^{am}_a}{2} + H_3^{am} & \frac{\theta^{am}_a}{2} + H_4^{am} \\
\frac{\theta^{am}_a}{2} & \frac{\theta^{am}_a}{2} + H_2^{am} & \frac{\theta^{am}_a}{2} + H_3^{am} & \frac{\theta^{am}_a}{2} + H_4^{am} & \frac{\theta^{am}_a}{2} + H_5^{am} \\
\frac{\theta^{am}_a}{2} & \frac{\theta^{am}_a}{2} + H_4^{am} & \frac{\theta^{am}_a}{2} + H_5^{am} & \frac{\theta^{am}_a}{2} + H_4^{am} & 0 \\
0 & 0 & 0 & 0 & 0
\end{vmatrix}
= -\frac{1}{4} \times H_4^{am} \times H_5^{am} \times \theta^{am}_a \left( H_1^{am} \theta^{am}_a - 4H_1^{am}H_3^{am} + 4(H_2^{am})^2 \right) > 0
\]

Thus, the conditions for matrix $\mathbf{B}^{am}$ to be positive definite is

\[
\theta^{am}_a = \frac{4t_a b}{\alpha_a} \left( \frac{f_{a,n}^{am} + f_{a,m}^{am} + f_{a,h}^{am}}{\alpha_a} \right)^3 < \frac{4H_1^{am}H_3^{am} - 4(H_2^{am})^2}{H_1^{am}}
\]

From $\frac{f_{a,n}^{am} + f_{a,m}^{am} + f_{a,h}^{am}}{\alpha_a} \leq \sum_{k \in K} D_k$ for all $a \in A_0$ we need to have that

\[
\frac{4t_a b}{\alpha_a} \left( \sum_{k \in K} D_k \right)^3 < \frac{4H_1^{am}H_3^{am} - 4(H_2^{am})^2}{H_1^{am}} \forall a \in A_0
\]

(ii) When $\theta^{am}_a = 0$ for some $a$. With $\theta^{am}_a = 0$, the matrix $\mathbf{\bar{B}}^{am}$ can be written as follows:
\[
\bar{B}^{am} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & H_1^{am} & H_2^{am} & 0 & 0 \\
0 & H_2^{am} & H_3^{am} & 0 & 0 \\
0 & 0 & 0 & H_4^{am} & 0 \\
0 & 0 & 0 & 0 & H_5^{am}
\end{bmatrix}
\]

Since \( \det(\bar{B}^{am}) = 0 \), under the conditions that \( \sum_{a \in A} D_{ka} a^{am} \times \left( \sum_{a \in A} D_{ka} a^{am} \right)^3 \leq \frac{4H_1^{am}H_2^{am} - 4H_2^{am}^2}{H_1^{am}} \forall a \in A_0 \), from the proof of (i) we know that the matrix \( \bar{B}^{am} \) is positive semidefinite but not positive definite. In this situation, there is no hope for global uniqueness of arc flows. Instead, we try to achieve the second best property, which is to derive local uniqueness of arc flows.

Let \( H_2' \triangleq \partial_{t \in \mathbb{R}} f_0^{am} - \kappa^{am} \partial_{t \in \mathbb{R}} E^{am} \) and \( H_2'' \triangleq -\partial_{t \in \mathbb{R}} S^{am} \). When \( \theta_a^{am} = 0 \), the matrix \( B^{am} \) can be written as follows:

\[
B^{am} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & H_1 & H_2' & 0 & 0 \\
0 & H_2'' & H_3 & 0 & 0 \\
0 & 0 & 0 & H_4 & 0 \\
0 & 0 & 0 & 0 & H_5
\end{bmatrix}
\]

When \( \theta_a^{am} = 0 \), we must have that \( f_0^{am} = f^{am} = f^{am} = f^{am} = 0 \). Let \( f^{am} \triangleq \begin{bmatrix} 0 & 0 & 0 & f^{am}_{a \in \mathbb{R}} \end{bmatrix} \).

Since the arc cost function (10) is convex, it is locally lipschitz at the point \( f_0^{am} \). Since the arc cost function (10) is also directional derivative at the point \( f_0^{am} \), by definition 3.1.2 of Facchinei and Pang (2003), it is B-differentiable at the point \( f_0^{am} \). Thus, Proposition 3.3.7 of Facchinei and Pang (2003) applies.

According to Proposition 3.3.7(a) of Facchinei and Pang (2003), to show a given solution \( f^{am} \) is locally unique, we need to show that the following homogeneous Complementarity Problem (CP) has \( f^{am} = 0 \) as the unique solution:

\[
C(f_0^{am}, \mathcal{H}, F', \Phi') \ni f^{am} \perp JF_r^{am}(f_0^{am})f^{am} \in C(f_0^{am}, \mathcal{H}, F', \Phi')^\ast
\]

where \( C(f_0^{am}, \mathcal{H}, F', \Phi') \) is a critical cone and \( C(f_0^{am}, \mathcal{H}, F', \Phi')^\ast \) represents its dual cone. From \( (f^{am})^T JF_r^{am}(f_0^{am})f^{am} = 0 \) we have that

\[
(f^{am})^T B^{am} f^{am} = 0
\]

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Under the conditions of Proposition 1, the $4 \times 4$ matrix in the middle above is positive definite. Thus we have that

$$f_{am} = f_{arp} = f_{ahp} = f_{atp} = 0$$

From $f_{am} \in C(f_0^{am}, \mathcal{H}, \Phi')$, by definition we have

$$f_{am} tt_{am}(f_0^{am}) + f_{am} tt_{am}(f_0^{am}) + f_{am} tt_{am}(f_0^{am}) + f_{am} tt_{am}(f_0^{am}) + f_{am} tt_{am}(f_0^{am}) = 0$$

Since $f_{am} = f_{am} = f_{am} = f_{am} = 0$ we must have

$$f_{am} tt_{am}(f_0^{am}) = 0$$

$$\Rightarrow f_{am} = 0$$

Thus, $f^{am} = 0$ is the unique solution for the homogeneous CP above-mentioned. And we have locally unique arc flows when $\theta^{am} = 0$ for some $a \in \mathcal{A}_0$. 

\[\square\]
Appendix 2. About path flows derived from the path-based model.

Since the path flows derived from the path-based model may not be unique, it is possible that after solving the path-based model, we will obtain a solution in which a rideshare passenger is served by multiple rideshare drivers along his/her path. In that case, the rideshare passenger may transfer between different rideshare vehicles. In Proposition 2 below, we show that given a feasible equilibrium solution of path flows with the transfers of rideshare passengers, we can construct a feasible equilibrium solution of path flows without transfers based on it. And the arc flows derived from these two path flow solutions will be the same, which can also be obtained by solving the equivalent arc-based model.

Proposition 2. Given a feasible equilibrium solution of path flows with transfers of rideshare passengers, namely a rideshare passenger is served by multiple rideshare drivers along his/her path, we can always construct a feasible equilibrium solution of path flows without transfers from it. Moreover, these two path flow solutions correspond to the same arc flows at the aggregate level and thus the same equilibrium state.

Proof. The proof is by construction. Suppose that for a feasible equilibrium solution of path flows, there exists a path \( p_{rp} \) of the extended network in which a rideshare passenger can be served by multiple rideshare drivers along the path. Denote \( p_0 \) as the corresponding path in the original network consisting of arcs \( a_1, a_2, \ldots, a_n \in A_0 \). Then the rideshare-passenger arcs along the path \( p_{rp} \) in the extended network can be denoted as \( T_{rp}(a_1), T_{rp}(a_2), \ldots, T_{rp}(a_n) \in A_{rp} \). Denote \( z^1_{p_{rp}} \) as the flow of rideshare passengers that need transfers. Then along the path, there must be flows of rideshare drivers on each arc, denoted as \( z^1_{T_{rd}(a_1)}, z^1_{T_{rd}(a_2)}, \ldots, z^1_{T_{rd}(a_n)} \), to serve these rideshare passengers. Denote \( p_{rd} \) as the path of rideshare drivers corresponding to \( p_{rp} \), which consists of arcs \( T_{rd}(a_1), T_{rd}(a_2), \ldots, T_{rd}(a_n) \in A_{rd} \). Then we can construct a solution without transfers of rideshare passengers as follows:

\[
\begin{align*}
 z^2_{p_{rd}} &= \min\{z^1_{T_{rd}(a_1)}, z^1_{T_{rd}(a_2)}, \ldots, z^1_{T_{rd}(a_n)}\} \\
 z^2_{p_{rp}} &= z^1_{p_{rp}} - z^2_{p_{rd}} \\
 z^2_{T_{rp}(a_i)} &= z^2_{p_{rd}} \quad \forall i = 1, 2, \ldots, n \\
 z^2_{T_{rd}(a_i)} &= z^1_{T_{rd}(a_i)} - z^2_{p_{rd}} \quad \forall i = 1, 2, \ldots, n
\end{align*}
\]

Note that with the construction above, the arc flows of rideshare drivers and passengers will not change as the summation of path flows, and the total travel demand will remain the same. As a result, the constructed solution is feasible, e.g., the rideshare capacity constraint (12) will not be violated. Moreover, since the total cost for a traveler on a path is the summation of all arc costs along that path, and arc cost is a function of arc flows, the constructed solution shares the same travel cost and thus the same equilibrium state as the previous one. Based on the equivalence of the path-based model and the arc-based model in Remark (ii) of Section 2.8, the equilibrium solution and the corresponding arc flows can be derived by solving the arc-based model.
Appendix 3. The equivalent mixed complementarity formulation of the path-based VI model.

Based on Proposition 1.2.1 of Facchinei and Pang (2003), the equivalent mixed complementarity formulation of the path-based VI model defined in Section 2.8 can be written as follows:

\[
0 \leq z_p \perp TC_p(z) - \lambda_p(\eta_a) - u_k \geq 0, \quad \forall p \in P_k, \forall k \in K
\]

\[
0 \leq u_k \perp D_k = \sum_{p \in P_k} z_p \quad \forall k \in K
\]

\[
0 \leq \eta_{a^{+am}} \perp \begin{cases} \sum_{p \in P^{am}, a_{rd} \in p} z_p - \sum_{p \in P^{am}, a_{rp} \in p} z_p \geq 0 \\ \sum_{p \in P^{am}, a_{rd} \in p} z_p - \sum_{p \in P^{am}, a_{rp} \in p} z_p \geq 0 \end{cases} \quad \forall a \in A_0, a_{rd} = T_{rd}(a), \quad a_{rp} = T_{rp}(a)
\]

where

\[
\lambda_p(\eta_a) \triangleq \sum_{T_{rd}(a) \in P^{am} \cap A_{rd}} \left( M \eta_{a^{+am}} - \eta_{a^{-am}} \right) + \sum_{T_{rd}(a) \in P^{pm} \cap A_{rd}} \left( M \eta_{a^{-pm}} - \eta_{a^{+pm}} \right) + \sum_{T_{rp}(a) \in P^{pm} \cap A_{rp}} \left( \eta_{a^{+pm}} - \eta_{a^{-pm}} \right)
\]
Appendix 4. The equivalent mixed complementarity formulation of the arc-based VI model.

Based on Proposition 1.2.1 of Facchinei and Pang (2003), the equivalent mixed complementarity formulation of the arc-based VI model defined in Section 2.8 can be written as follows:

\[
0 \leq x_{a; k}^{am} + \eta_{T_a(a)} + \mu_i^{k; am} + \mu_j^{k; am} - \zeta_{a; k} \geq 0, \quad \forall a = (i, j) \in A_{id}, \forall k \in K
\]

\[
0 \leq x_{a; k}^{am} + \eta_{T_a(a)} - M \eta_{T_a(a)} - \mu_i^{k; am} + \mu_j^{k; am} - \zeta_{a; k} \geq 0, \quad \forall a = (i, j) \in A_{ad}, \forall k \in K
\]

\[
0 \leq x_{a; k}^{am} + \eta_{T_a(a)} + \mu_i^{k; am} + \mu_j^{k; am} \geq 0, \quad \forall a = (i, j) \in A_{hp}, \forall k \in K
\]

\[
0 \leq x_{a; k}^{am} + \eta_{T_a(a)} - \mu_i^{k; am} + \mu_j^{k; am} \geq 0, \quad \forall a = (i, j) \in A_{rp}, \forall k \in K
\]

\[
0 \leq x_{a; k}^{pm} + \eta_{T_a(a)} + \mu_i^{k; pm} + \mu_j^{k; pm} - \zeta_{a; k} \geq 0, \quad \forall a = (i, j) \in A_{ad}, \text{ associated } \tilde{k}
\]

\[
0 \leq x_{a; k}^{pm} + \eta_{T_a(a)} - M \eta_{T_a(a)} - \mu_i^{k; pm} + \mu_j^{k; pm} - \zeta_{a; k} \geq 0, \quad \forall a = (i, j) \in A_{ad}, \text{ associated } \tilde{k}
\]

\[
0 \leq x_{a; k}^{pm} + \eta_{T_a(a)} - \mu_i^{k; pm} + \mu_j^{k; pm} \geq 0, \quad \forall a = (i, j) \in A_{hp}, \text{ associated } \tilde{k}
\]

\[
0 \leq x_{a; k}^{pm} + \eta_{T_a(a)} - \mu_i^{k; pm} + \mu_j^{k; pm} \geq 0, \quad \forall a = (i, j) \in A_{rp}, \text{ associated } \tilde{k}
\]

\[
0 \leq \eta_{a}^{am} \perp \sum_{k \in K} x_{a; k}^{am} - \sum_{k \in K} x_{a; k}^{am} \geq 0
\]

\[
0 \leq \eta_{a}^{pm} \perp M \sum_{k \in K} x_{a; k}^{pm} - \sum_{k \in K} x_{a; k}^{pm} \geq 0
\]

\[
\mu_i^{k; am}, \mu_j^{k; pm} \text{ free } \perp \begin{cases} 
\sum_{a \in \text{IN}(d_k)} x_{a; k}^{am} = \sum_{a \in \text{OUT}(d_k)} x_{a; k}^{pm} = \sum_{a \in \text{IN}(d_k)} x_{a; k}^{am} = \sum_{a \in \text{OUT}(d_k)} x_{a; k}^{pm} = D_k \\
\forall k \in K \text{ and associated } \tilde{k}
\end{cases}
\]

\[
\sum_{a \in \text{IN}(i)} x_{a; k}^{am} = \sum_{a \in \text{OUT}(i)} x_{a; k}^{pm} = 0 \quad \forall i \in N \setminus \{o_k, d_k\}, \forall k \in K
\]

\[
\sum_{a \in \text{IN}(i)} x_{a; k}^{am} = \sum_{a \in \text{OUT}(i)} x_{a; k}^{pm} = 0 \quad \forall i \in N \setminus \{o_k, d_k\}, \text{ associated } \tilde{k}
\]

\[
\zeta_{a; k} \text{ free } \perp \sum_{a_{id}, a_{rd} \in \text{OUT}(o_k)} (x_{a_{id}}^{am} + x_{a_{rd}}^{am}) = \sum_{a_{id}, a_{rd} \in \text{IN}(d_k)} (x_{a_{id}}^{pm} + x_{a_{rd}}^{pm}) \quad \forall k \in K \text{ and associated } \tilde{k}
\]